Dynamics of Structures
A Modal Approach

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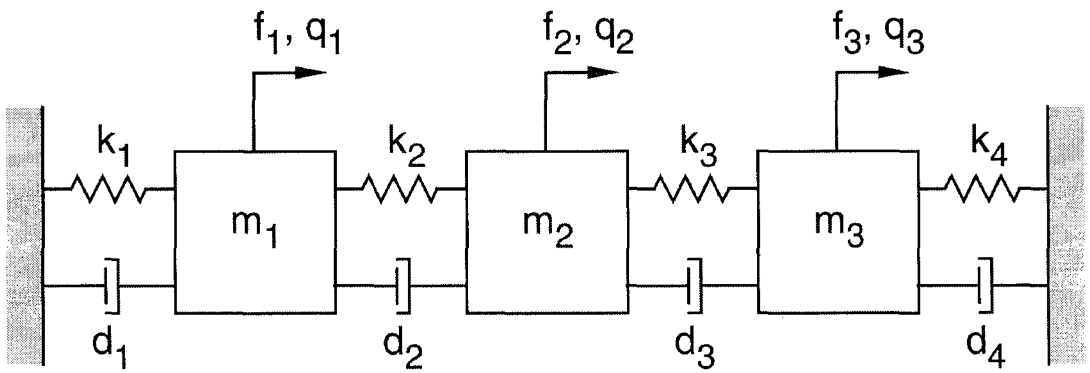
Part 1 Structures:

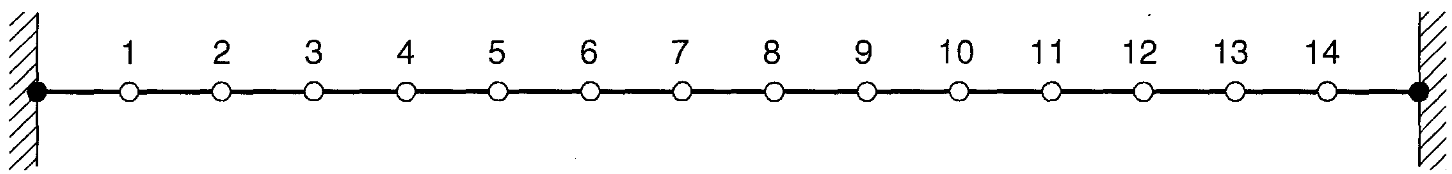
examples,
properties,
and models

Examples: a simple structure



Three-mass system:
not realistic, but useful in explanations

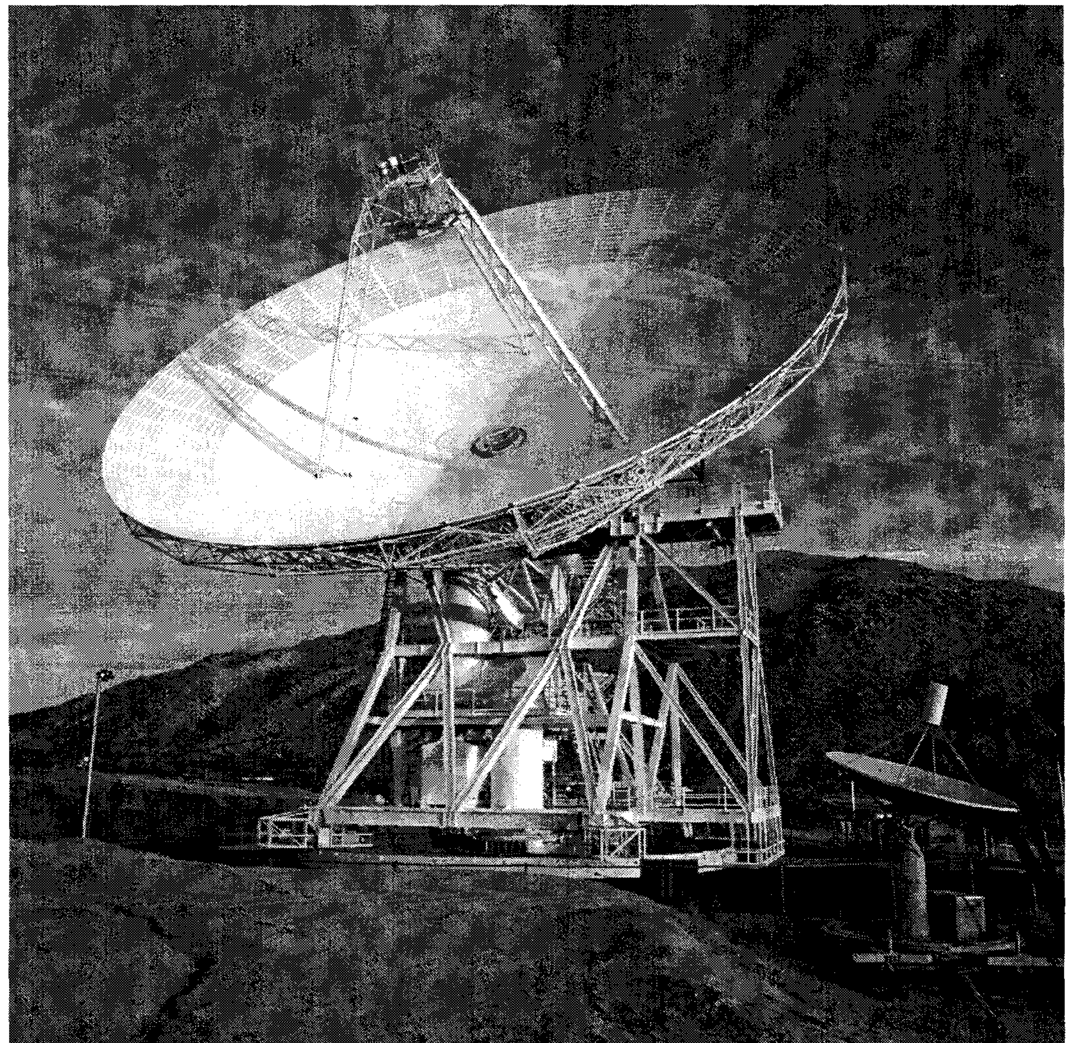
Examples: a beam



The finite element model of a clamped beam:

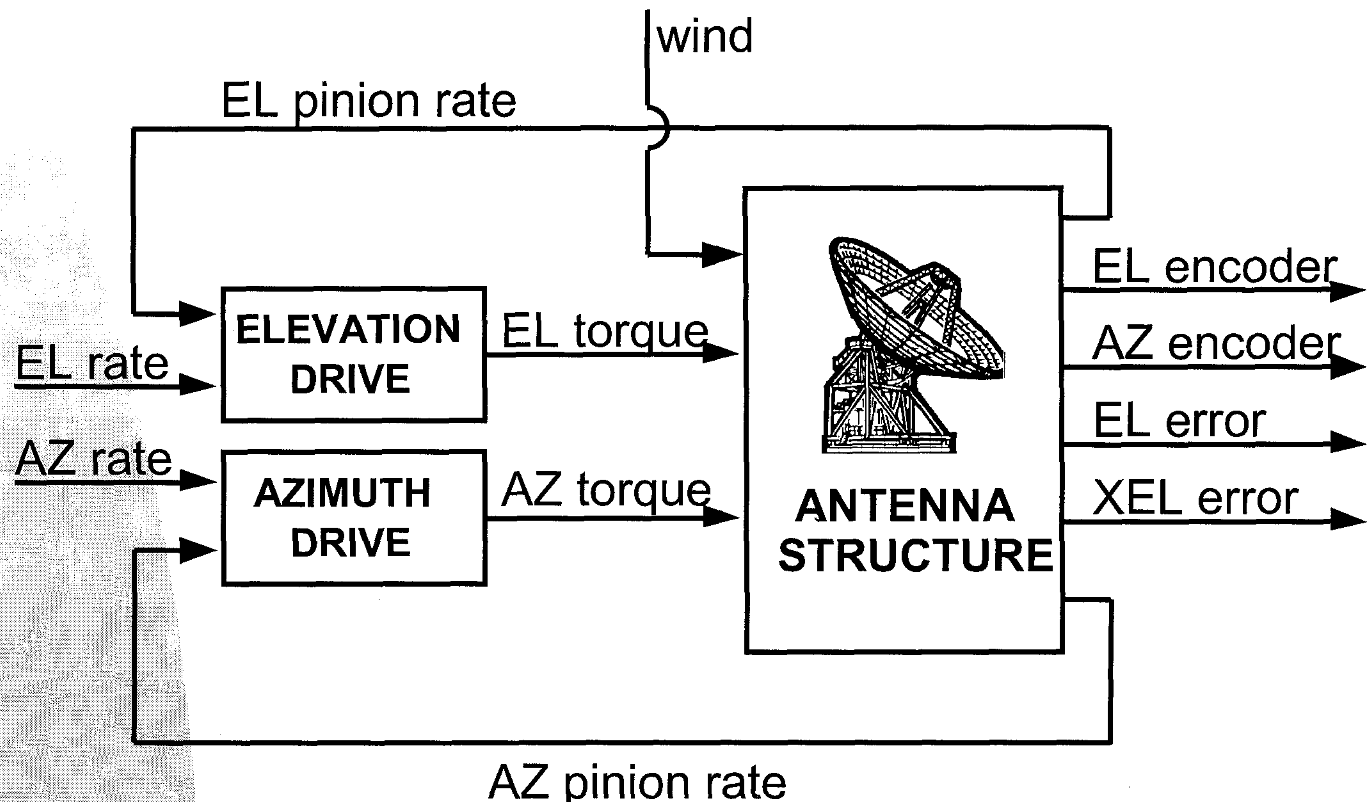
• more realistic and easy to visualize the properties

Examples: the DSN antenna

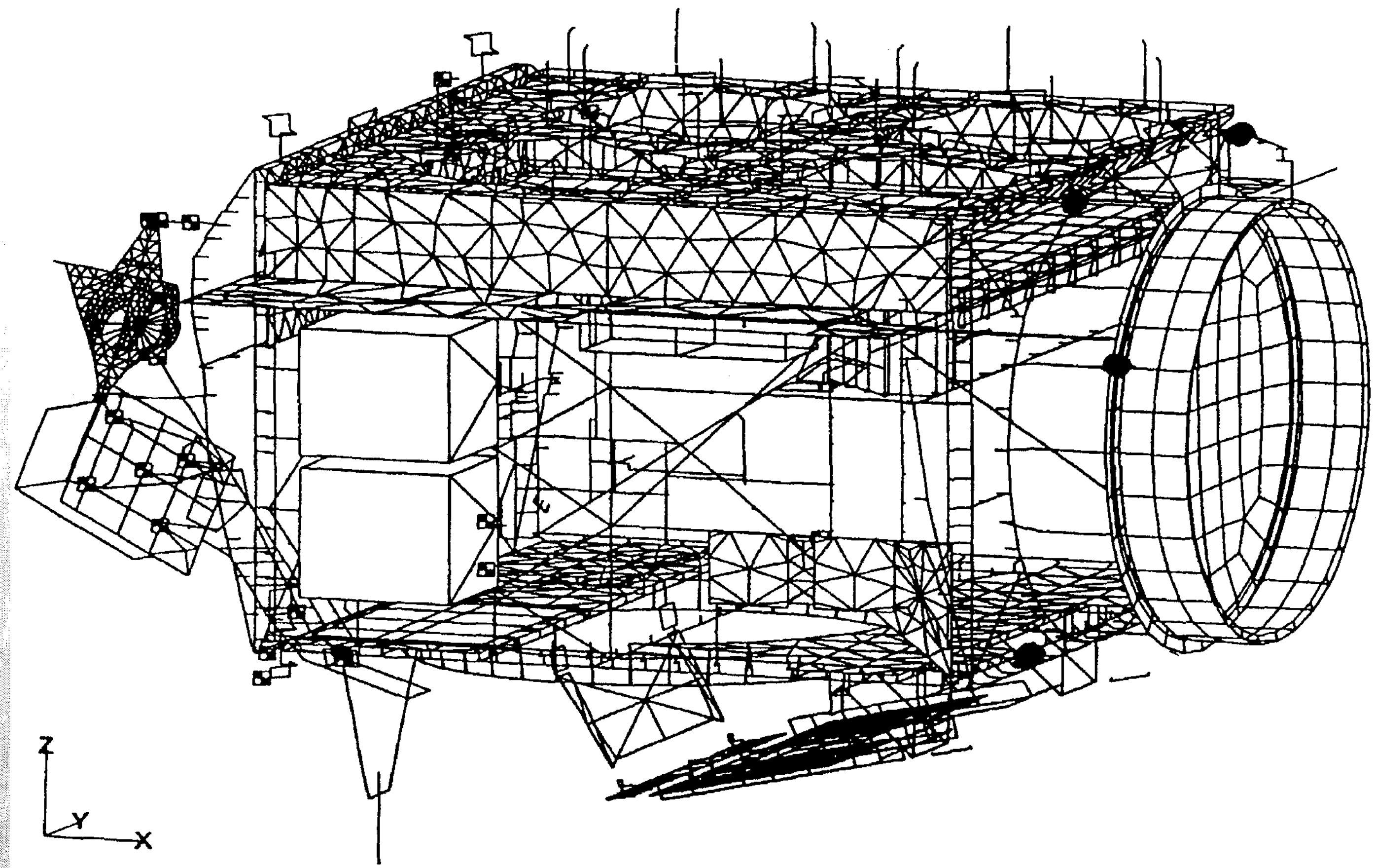


The DSN antenna (cont)

A combination of structure, mechanical (gearboxes), electrical (motors and tachometers). and electronics (circuits, filters and amplifiers) hardware



Examples: the finite element model of the International Space Station structure



Structure: definition

A structure is a linear system which is:

- finite-dimensional;
- controllable, and observable;
- its poles are complex with small real parts;
- its poles are non-clustered.

Structure: properties

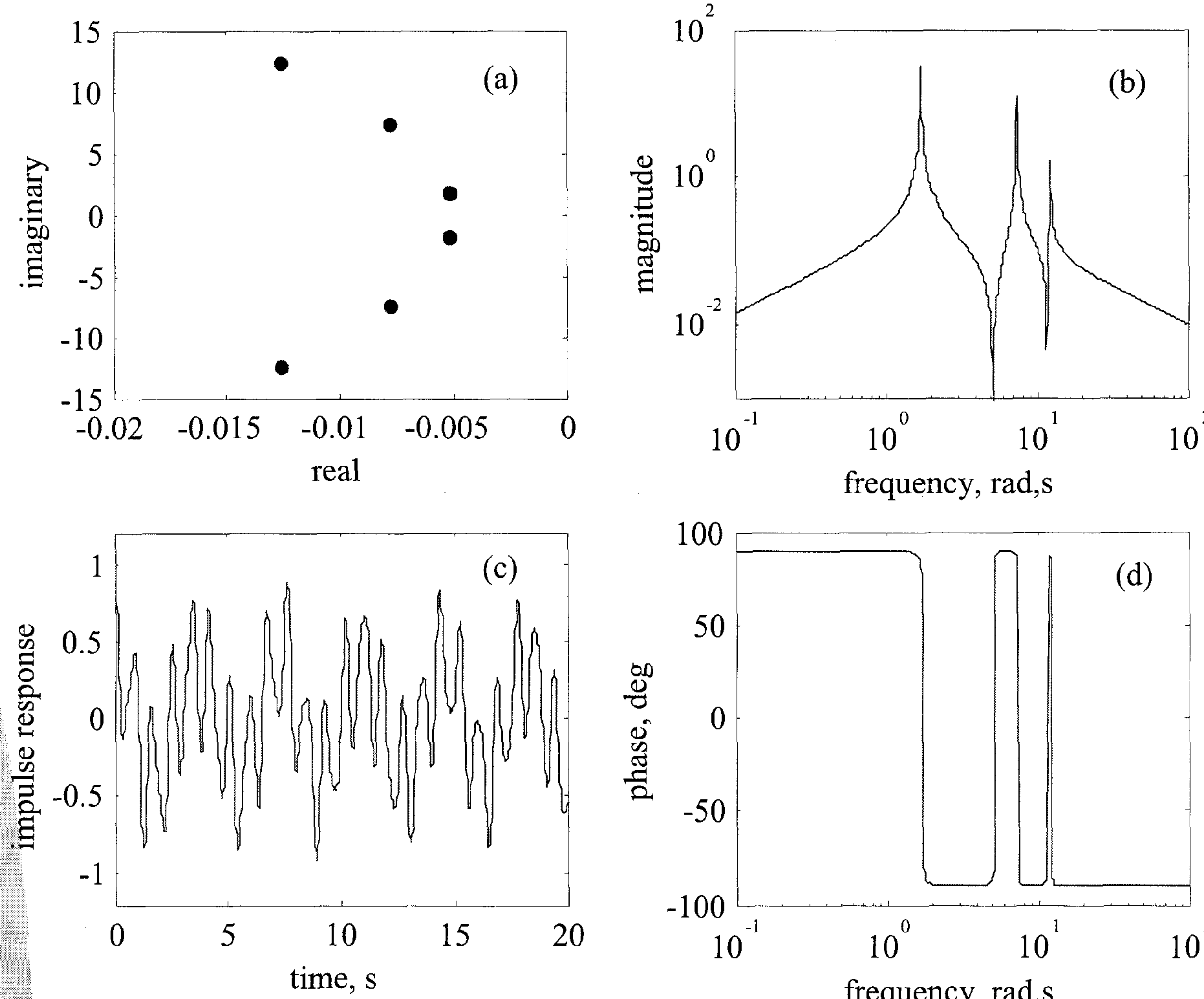
Properties of a typical flexible structure:

a) poles - complex;

b) magnitude of t.f.: resonance peaks;

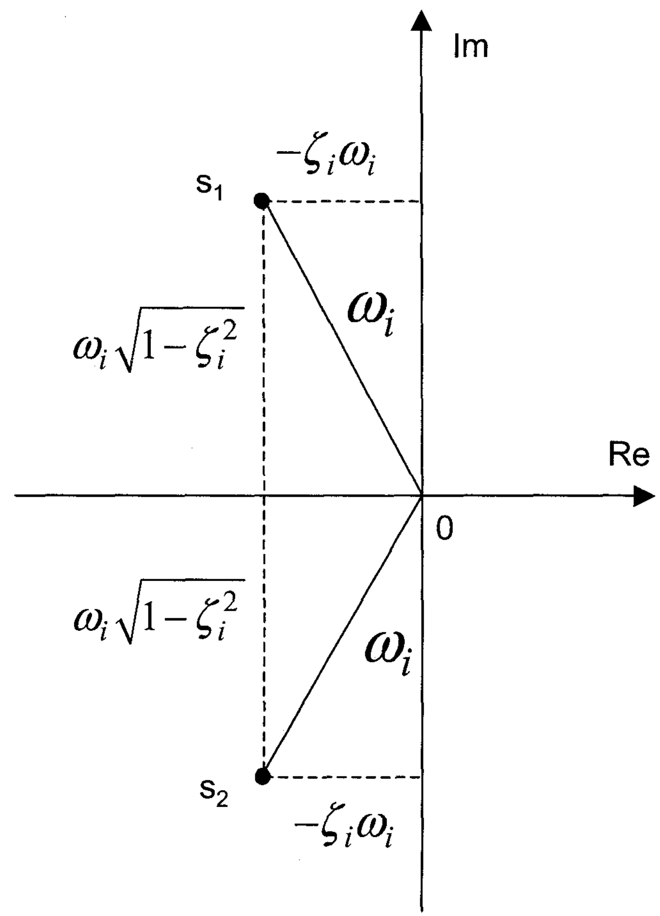
c) impulse response: several harmonics;

d) phase of a t.f.: 180 deg change at resonance freq.



Structure: poles

Complex,
small real parts



Structure: properties

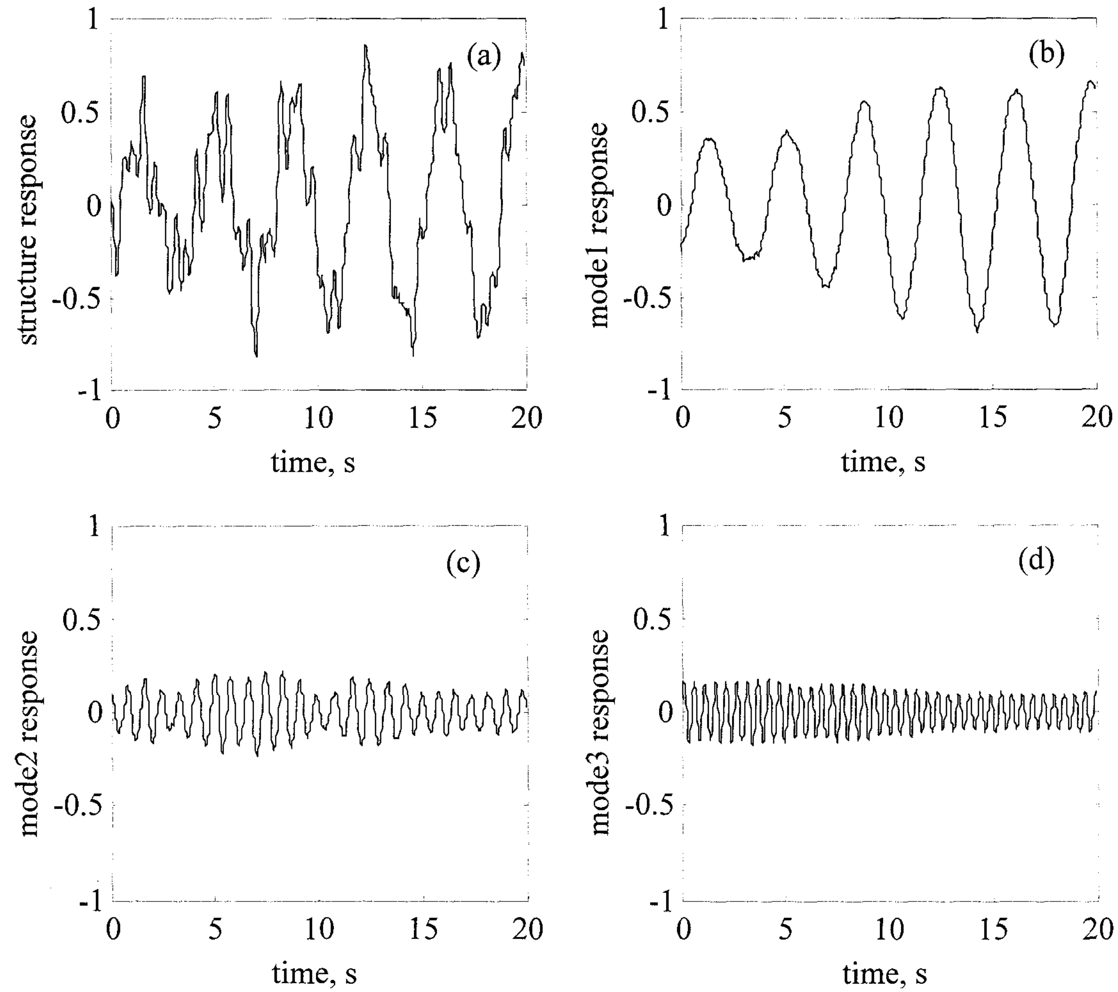
Response to the white noise input:

(a) total response is composed of three harmonics:,

(b) mode 1,

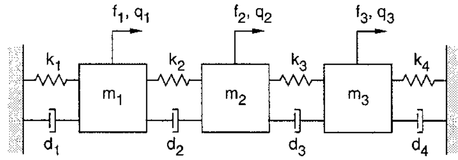
(c) mode 2,

(d) mode 3



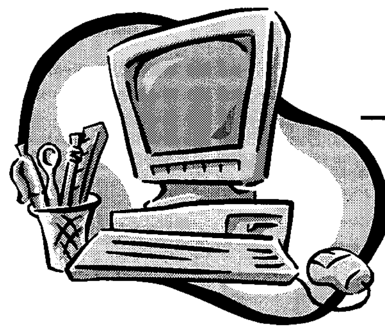
Models: three ways to obtain them

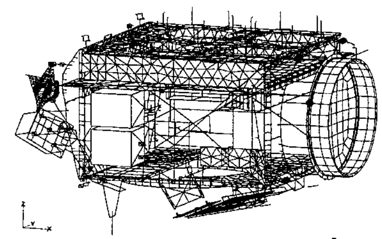
Physical laws (Lagrange, Newton, d'Alambert)



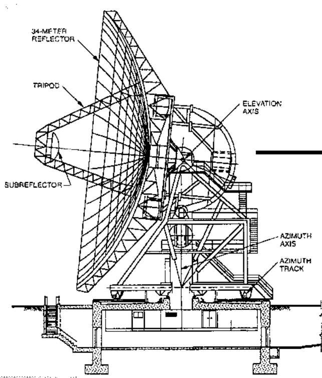
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right)-\frac{\partial T}{\partial q_k}+\frac{\partial V}{\partial q_k}=Q_{knc}$$

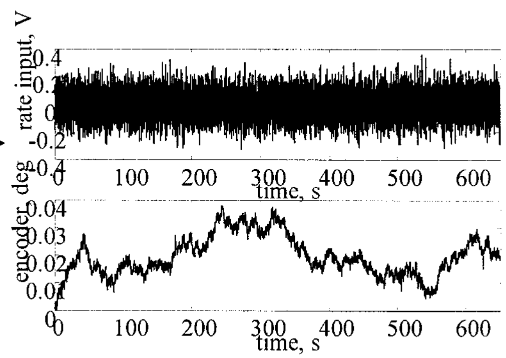
Finite element models





System identification





$$M\ddot{q}+D\dot{q}+Kq=B_o u$$
$$y=C_{oq}q+C_{ov}\dot{q}$$

Models of a linear system

State-space representation *(A,B,C)*:

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

Transfer function, *G(s)*:

$$y(s) = G(s)u(s)$$

$$G(s) = C(sI - A)^{-1}B$$

Second-order nodal model

$$M\ddot{q} + D\dot{q} + Kq = B_o u$$
$$y = C_{oq} q + C_{ov} \dot{q}$$

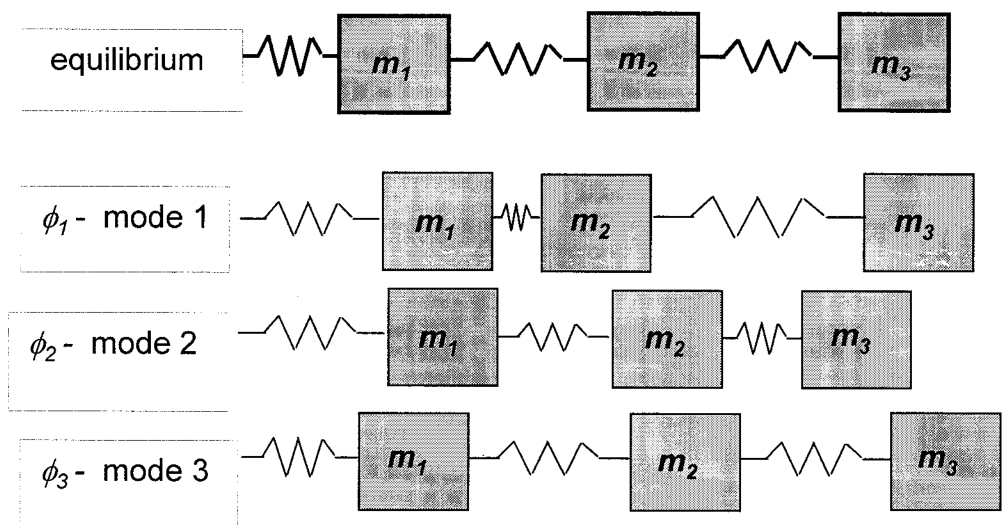
q	displacement vector
u	input vector,
y	output vector,
M	mass matrix, positive definite,
D	damping matrix, positive semidefinite,
K	stiffness matrix, positive semidefinite
B_o	input matrix
C_{oq}	output displacement matrix
C_{ov}	output velocity matrix

Second-order modal model

Free vibrations: natural frequencies and modes:

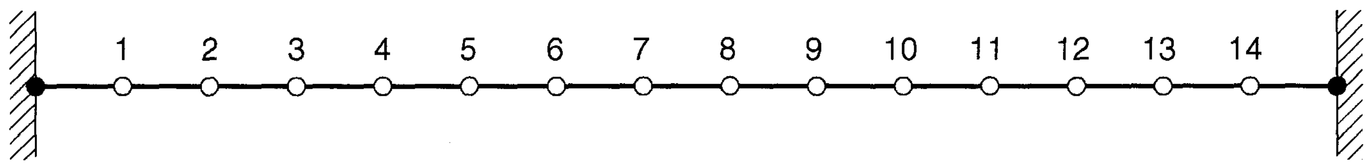
$$\boxed{\omega_i}$$

$$\boxed{\phi_i = \begin{Bmatrix} \phi_{i1} \\ \phi_{i2} \\ \vdots \\ \phi_{in} \end{Bmatrix}}$$

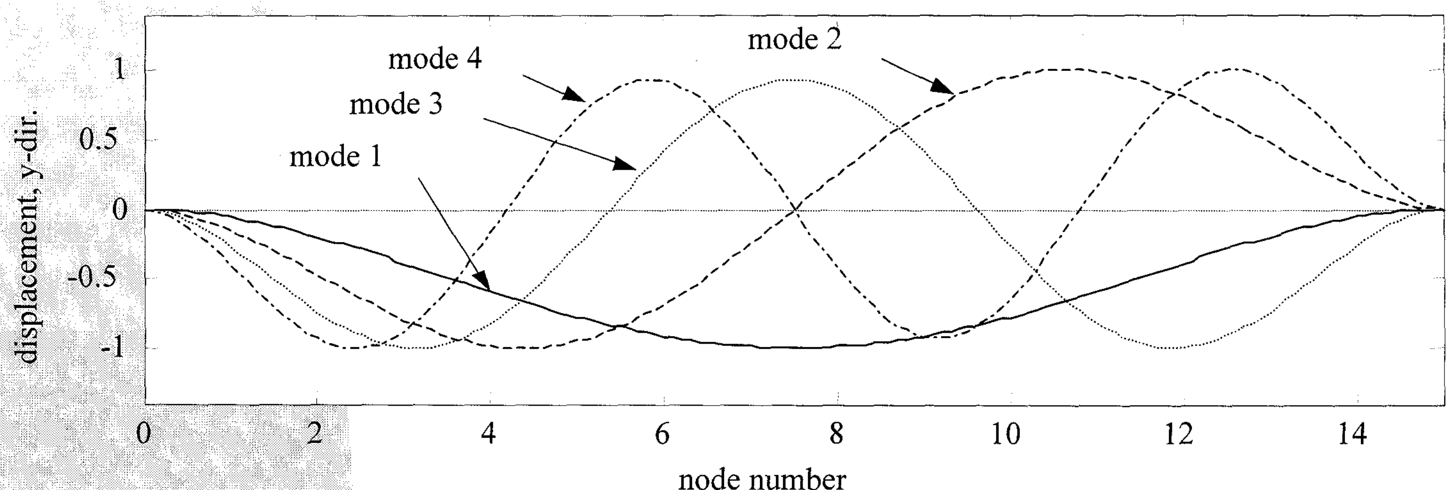


Natural modes

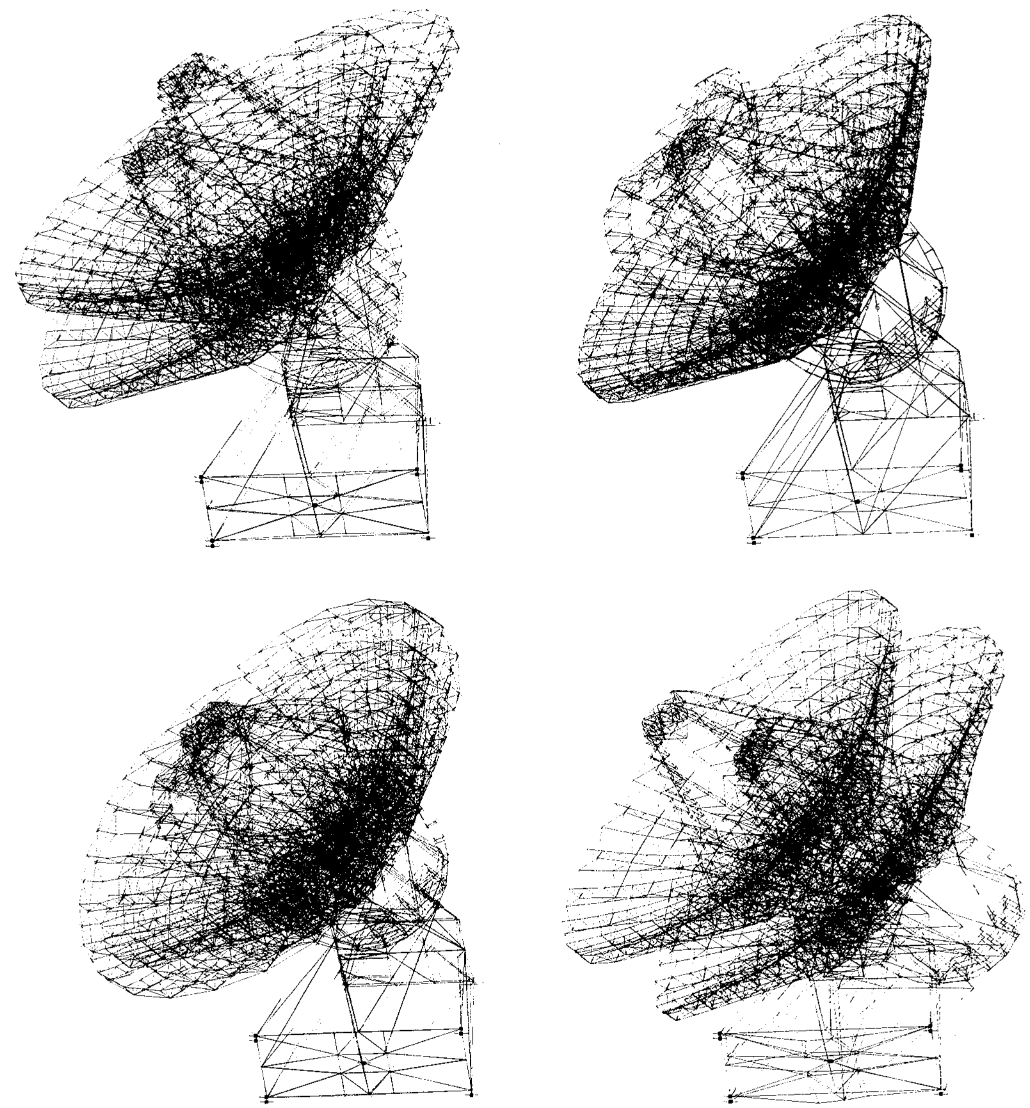
Beam:

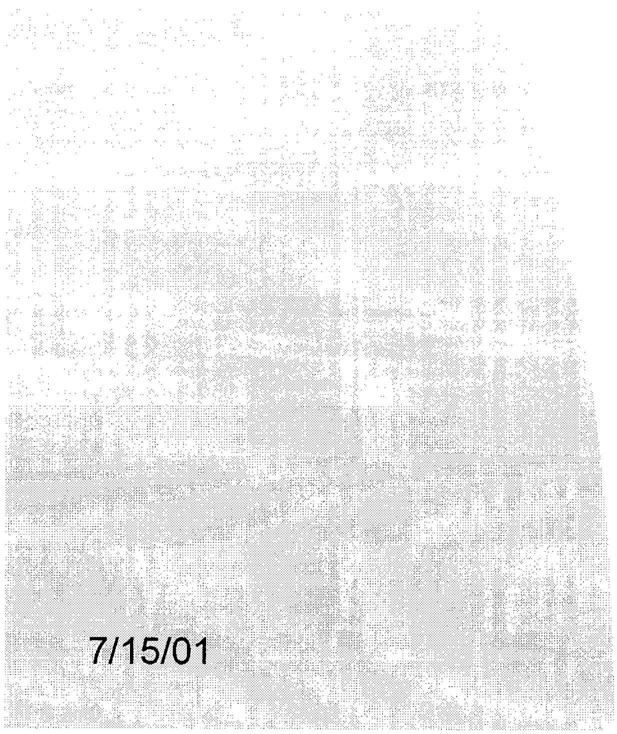


and its natural modes:



DSN
antenna
natural
modes:





Second-order modal model

Matrix of natural frequencies:

$$\Omega = \begin{bmatrix} \omega_1 & 0 & \cdots & 0 \\ 0 & \omega_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \omega_n \end{bmatrix}$$

and modal matrix:

$$\Phi = \begin{bmatrix} \phi_1 & \phi_2 & \dots & \phi_n \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{21} & \cdots & \phi_{n1} \\ \phi_{12} & \phi_{22} & \cdots & \phi_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ \phi_{1n_d} & \phi_{2n_d} & \cdots & \phi_{nn_d} \end{bmatrix}$$

Second-order modal model

Modal transformation:

$$q = \Phi q_m$$

produces diagonal modal matrices:

$$M_m = \Phi^T M \Phi$$

$$K_m = \Phi^T K \Phi$$

$$D_m = \Phi^T D \Phi$$

Second-order modal model

In modal coordinates

$$\boxed{\begin{aligned}&\ddot{q}_m + 2Z\Omega\dot{q}_m + \Omega^2 q_m = B_m u\\ &y = C_{mq} q_m + C_{mv}\dot{q}_m\end{aligned}}$$

or

$$\boxed{\begin{aligned}&\ddot{q}_{mi} + 2\zeta_i\omega_i\dot{q}_{mi} + \omega_i^2 q_{mi} = b_{mi} u\\ &y_i = c_{mqi} q_{mi} + c_{mvi}\dot{q}_{mi}\\ &y = \sum_{i=1}^{n} y_i\end{aligned}} \qquad i = 1, \ldots, n,$$

State-space structural models

Nodal model:

$$x = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} q \\ \dot{q} \end{Bmatrix} \longleftarrow$$ state vector

state space representation (A,B,C)

$$A = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ M^{-1}B_o \end{bmatrix},$$

$$C = \begin{bmatrix} C_{oq} & C_{ov} \end{bmatrix}$$

State-space models in modal coordinates

$$x = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} q_m \\ \dot{q}_m \end{Bmatrix}$$

state vector

state space representation (A,B,C)

$$A = \begin{bmatrix} 0 & I \\ -\Omega^2 & -2Z\Omega \end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ B_m \end{bmatrix},$$

$$C = \begin{bmatrix} C_{mq} & C_{mv} \end{bmatrix}$$

State-space modal models

In this model the state matrix is in the following form ($\times$ denotes a non-zero element)

$$A_m = \mathrm{diag}(A_{mi}) = \begin{bmatrix} \times & \times & 0 & 0 & \cdots & \cdots & 0 & 0 \\ \times & \times & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & \times & \times & \cdots & \cdots & 0 & 0 \\ 0 & 0 & \times & \times & \cdots & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \cdots & \times & \times \\ 0 & 0 & 0 & 0 & \cdots & \cdots & \times & \times \end{bmatrix}$$

State-space modal model

$$\dot{x}_i = A_{mi} x_i + B_{mi} u$$
$$y_i = C_{mi} x_i$$
$$y = \sum_{i=1}^{n} y_i$$

state equations

$$A_{mi} = \begin{bmatrix} 0 & \omega_i \\ -\omega_i & -2\zeta_i\omega_i \end{bmatrix},$$
$$B_{mi} = \begin{bmatrix} 0 \\ b_{mi} \end{bmatrix},$$
$$C_{mi} = \begin{bmatrix} \dfrac{c_{mqi}}{\omega_i} & c_{mvi} \end{bmatrix}$$

state space representation (*A*,*B*,*C*)

Modal vector

$$x = \begin{Bmatrix} x_1 \\ \boxed{x_2} \\ \vdots \\ x_n \end{Bmatrix}$$

modal component

$$x_2 = \begin{Bmatrix} \omega_2 q_{m2} \\ \dot{q}_{m2} \end{Bmatrix}$$

Transfer function in modal coordinates

The structural transfer function is a sum of modal t.f.

$$\boxed{G(\omega) = \sum_{i=1}^{n} G_{mi}(\omega)}$$

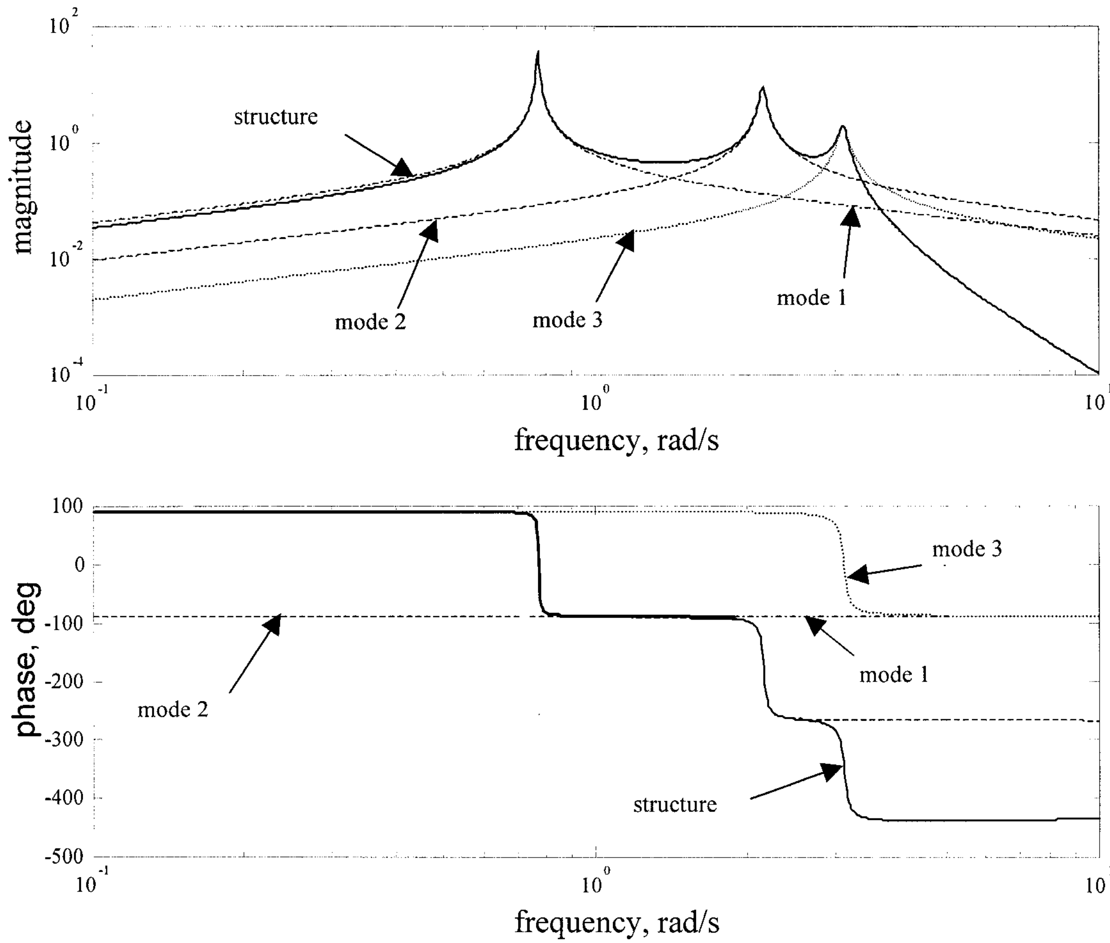
$$G_{mi}(\omega) = \frac{(c_{mqi} + j\omega c_{mvi})b_{mi}}{\omega_i^2 - \omega^2 + 2j\zeta_i\omega_i\omega}$$

Decomposition of the t. f.

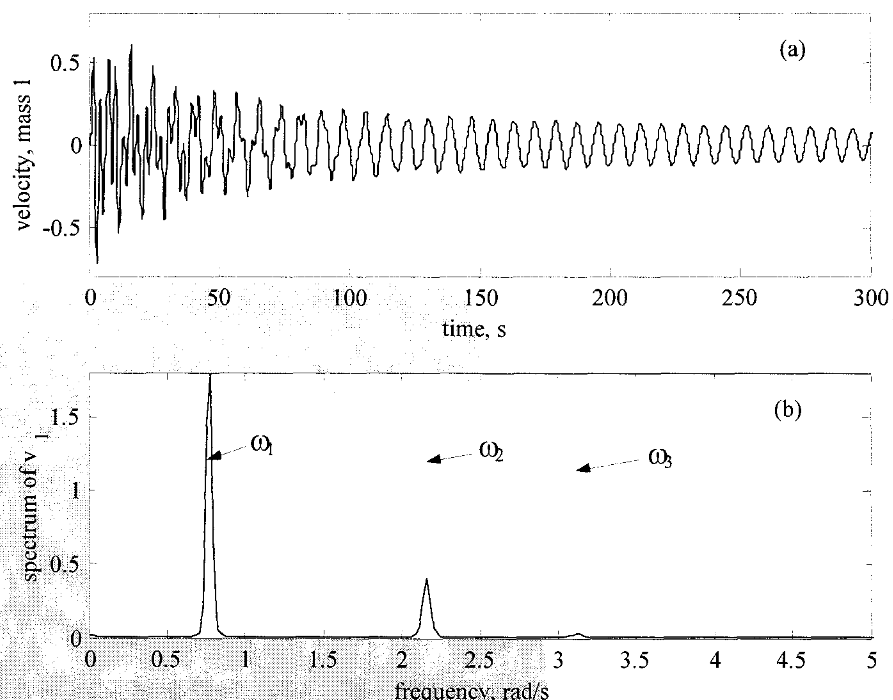
(in modal coordinates)

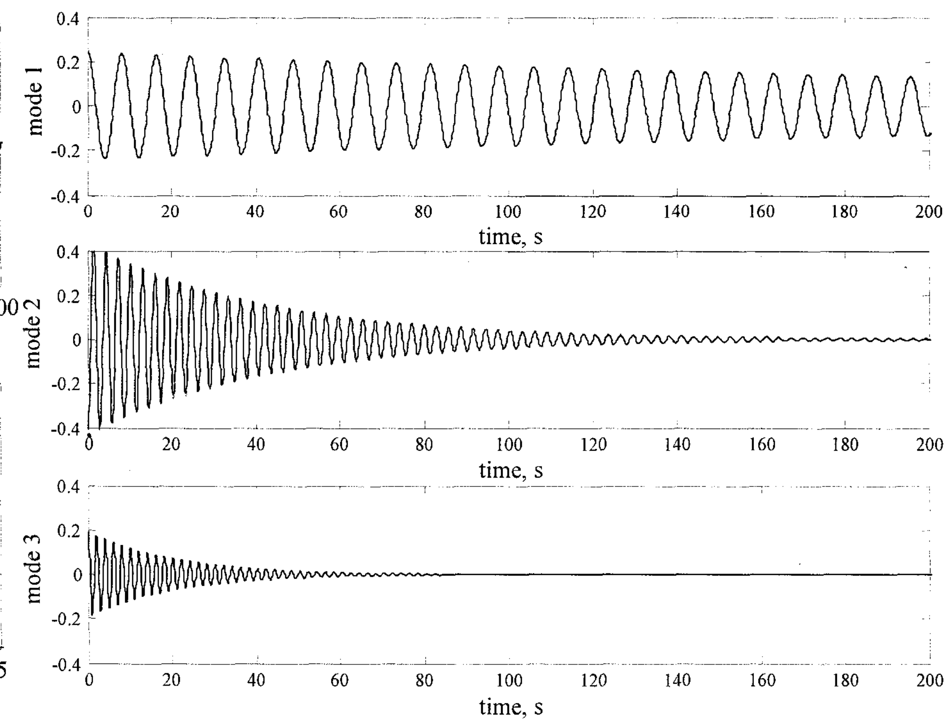
$$G(\omega) = \sum_{i=1}^{n} G_{mi}(\omega) =$$

$$\sum_{i=1}^{n} \frac{(c_{mqi} + j\omega c_{mvi}) b_{mi}}{\omega_i^2 - \omega^2 + 2j\zeta_i \omega_i \omega}$$



Decomposition in time domain

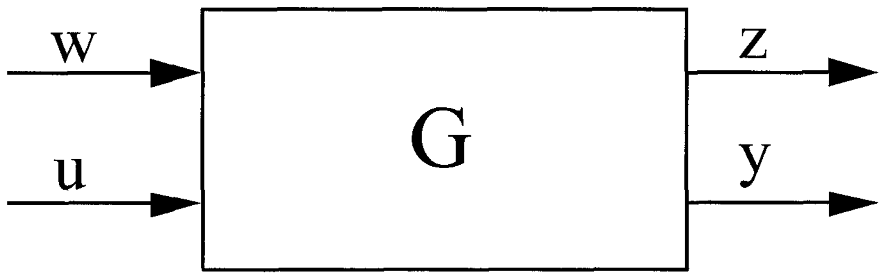




(in modal coordinates)

$$h(t)=\sum_{i=1}^{n} h_i(t)$$

Generalized model



<u>The inputs to the generalized model consist of two vector signals:</u>

- The actuator vector, denoted *u*, which consists of all inputs handled by the controller, or applied as test inputs.
- The disturbance vector, *w*, noises and disturbances, which are not manipulated by the controller, or are not a part of the test input.

<u>The outputs of the generalized model consist of two vector signals:</u>

- The sensor vector, *y*, used for the controller for the feedback purposes, or the measured test signals,
- The performance vector, *z*, the outputs to be controlled, or to evaluate test performance.

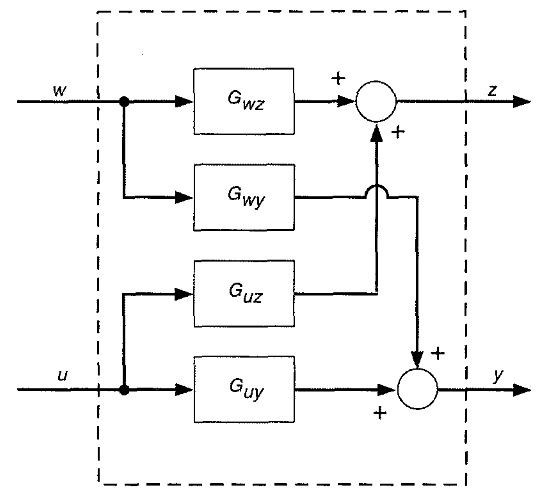
Generalized model

State-space representation:

$$\dot{x} = Ax + B_u u + B_w w$$

$$y = C_y x$$

$$z = C_z z$$



Transfer function:

$$y = G_{uy} u + G_{wy} w$$

$$z = G_{uz} u + G_{wz} w$$

$$G_{uy} = C_y (sI - A)^{-1} B_u$$

$$G_{wy} = C_y (sI - A)^{-1} B_w$$

$$G_{uz} = C_z (sI - A)^{-1} B_u$$

$$G_{wz} = C_z (sI - A)^{-1} B_w$$

Part 2
Structural norms

- System norms indicate the intensity of its response to standard excitation.
- H_2, H_∞, and Hankel norms are discussed
- The norms are used in the model reduction and in the actuator/sensor placement procedures.

H_2 norm

Definition: for SISO systems

• integral of the square of magnitude of t.f. (rms response to white noise)

$$\|G\|_2^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{tr}(G^*(\omega)G(\omega))d\omega$$

•rms impulse response

$$\|G\|_2^2 = \|g(t)\|_2^2 = \int_0^{\infty} \mathrm{tr}(g^T(t)g(t))dt$$

H_2 norm

Calculation

$$\|G\|_2 = \sqrt{\mathrm{tr}(C^T C W_c)}$$

$$\|G\|_2 = \sqrt{\mathrm{tr}(B B^T W_o)}$$

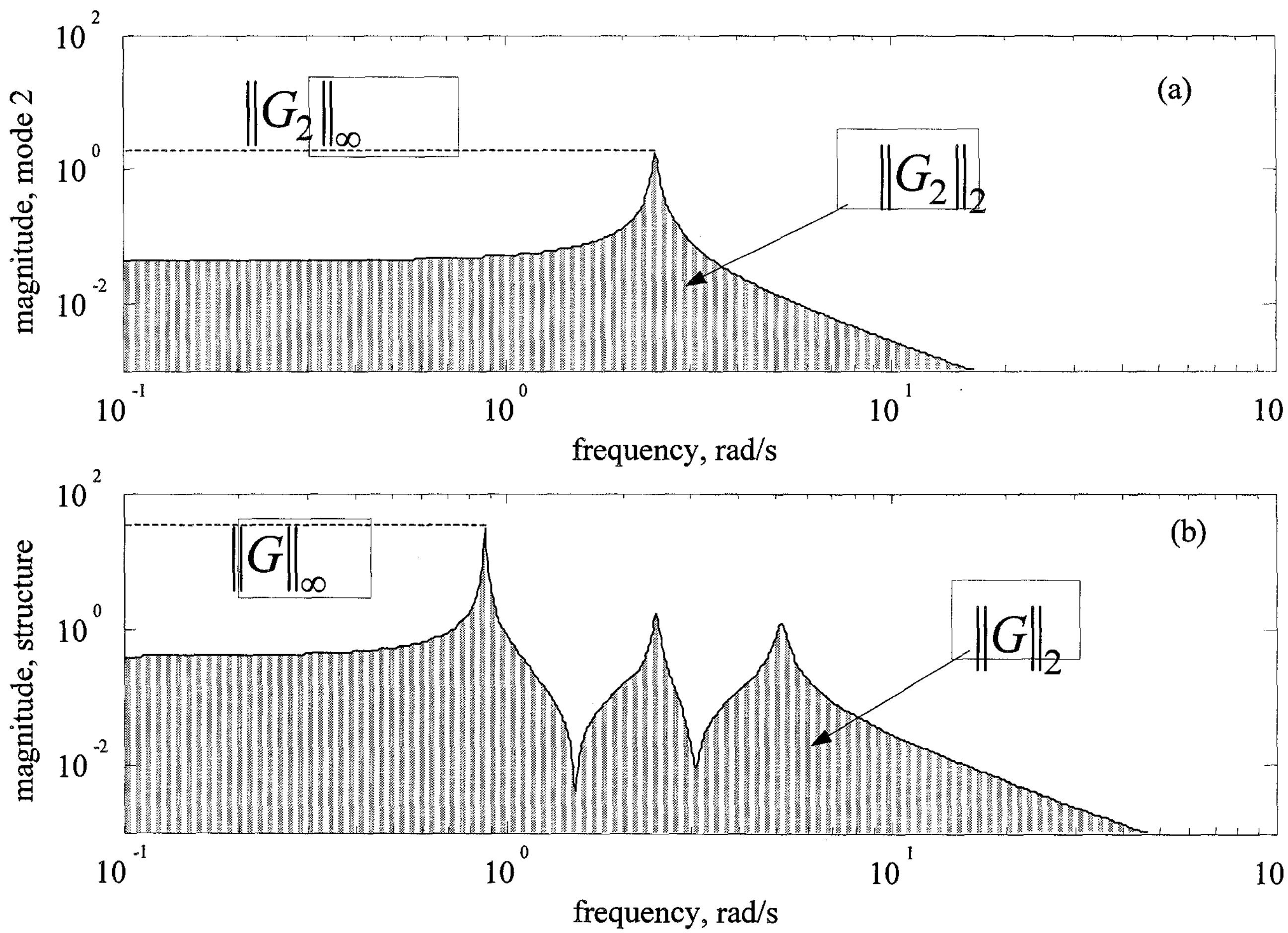
$W_c, \quad W_o$ are the controllability and observability grammians obtained from the following Lyapunov equations

$$A W_c + W_c A^T + B B^T = 0$$

$$A^T W_o + W_o A + C^T C = 0$$

H_2 norm

Interpretation:



H_∞ norm

Definition:

$$\|G\|_\infty = \max_\omega \sigma_{\max}(G(\omega))$$

Application:

$$\|y\|_2 \leq \|G\|_\infty \|u\|_2$$

Determination (as max ρ):

$$A^T S + SA + \rho^{-1} SBB^T S + \rho^{-1} C^T C = 0$$

Hankel norm

Definition:

$$\|G\|_h = \sup \frac{\|y(t)\|_2}{\|u(t)\|_2} \quad \text{where} \quad \begin{cases} u(t) = 0 & \text{for } t > 0 \\ y(t) = 0 & \text{for } t < 0 \end{cases}$$

Determination:

$$\|G\|_h = \sqrt{\lambda_{\max}(W_c W_o)}$$

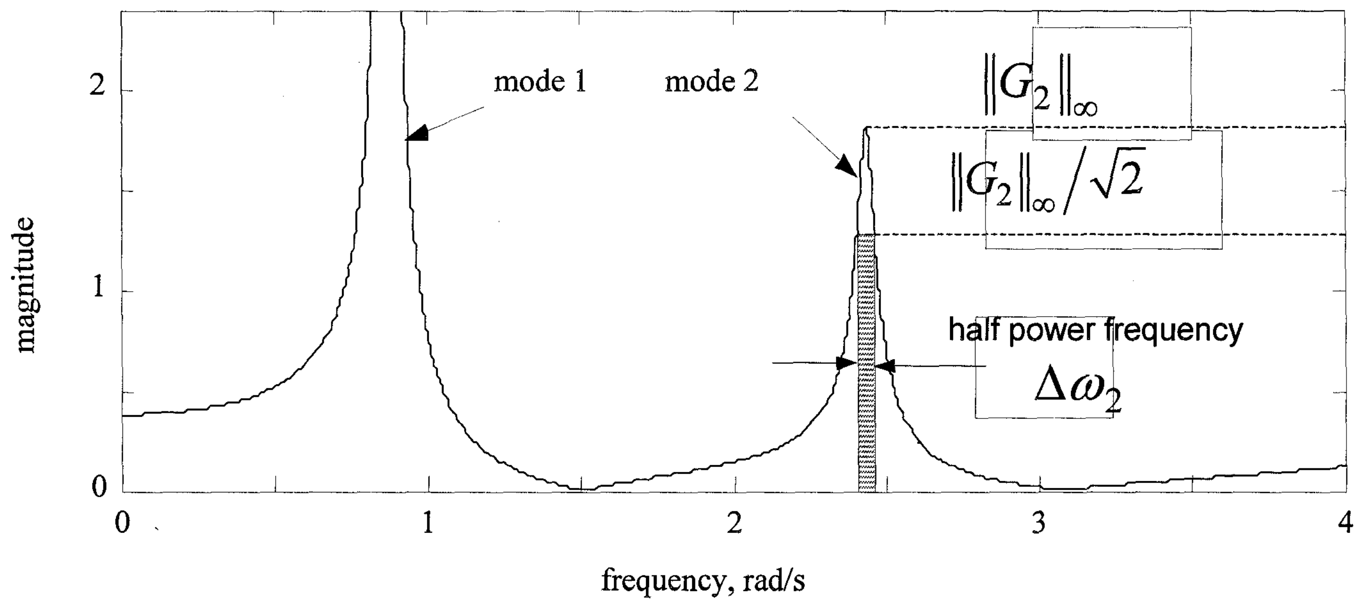
Relationship:

$$\|G\|_h \leq \|G\|_\infty$$

$$\|G\|_\infty = \sup_{u(t)\neq 0} \frac{\|y(t)\|_2}{\|u(t)\|_2}$$

Norms of a single mode

Determination of norms of a single mode:



Norms of a single mode

$$\|G_i\|_2 \cong \frac{\|B_i\|_2 \|C_i\|_2}{2\sqrt{\zeta_i \omega_i}}$$

$$\|G_i\|_\infty \cong \frac{\|B_i\|_2 \|C_i\|_2}{2\zeta_i \omega_i}$$

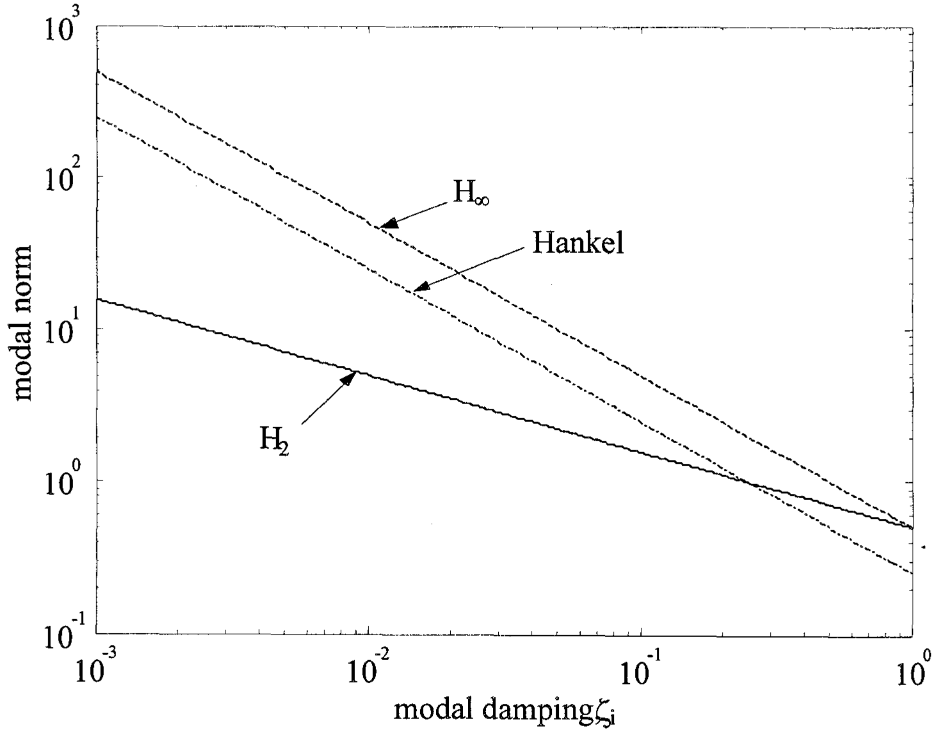
$$\|G_i\|_h \cong \frac{\|B_i\|_2 \|C_i\|_2}{4\zeta_i \omega_i}$$

Relationship between H_2, H_∞, and Hankel norms

$$\|G_i\|_\infty \cong 2\|G_i\|_h \cong \sqrt{\zeta_i \omega_i}\,\|G_i\|_2$$

Norms of a single mode

Relationship between H_2, H_∞, and Hankel norms

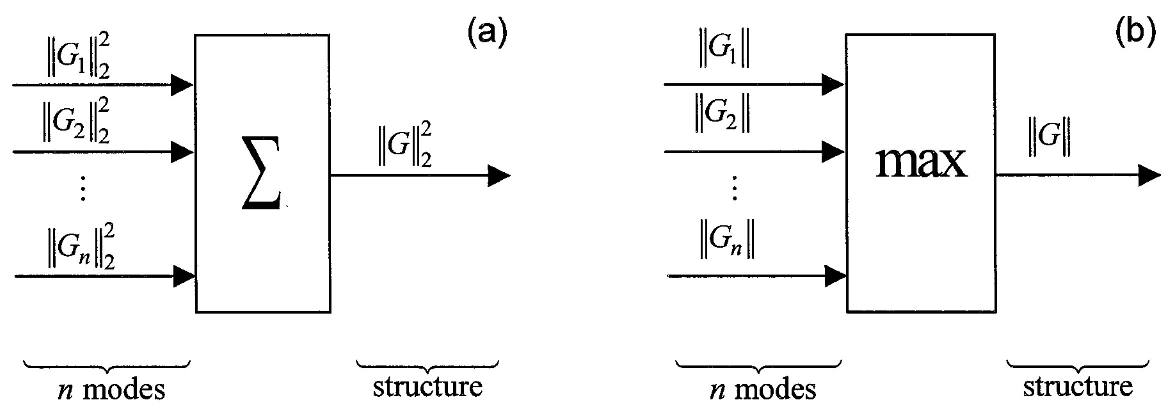


Norms of a structure

$$\|G\|_2 \cong \sqrt{\sum_{i=1}^{n} \|G_i\|_2^2}$$

$$\|G\|_\infty \cong \max_i \|G_i\|_\infty$$

$$\|G\|_h \cong \max_i \|G_i\|_h$$



Norms of a mode with actuators

The H_2, H_∞, and Hankel norms of the ith mode of a structure with a set of s actuators is the rms sum of norms of the mode with each single actuator from this set, i.e.,

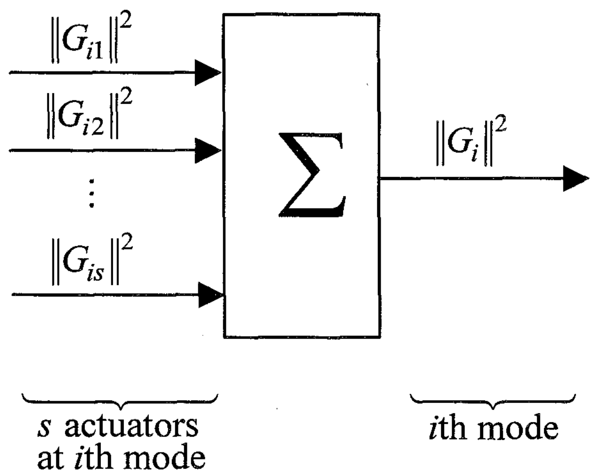
$$\|G_i\| \cong \sqrt{\sum_{j=1}^{s} \|G_{ij}\|^2}$$

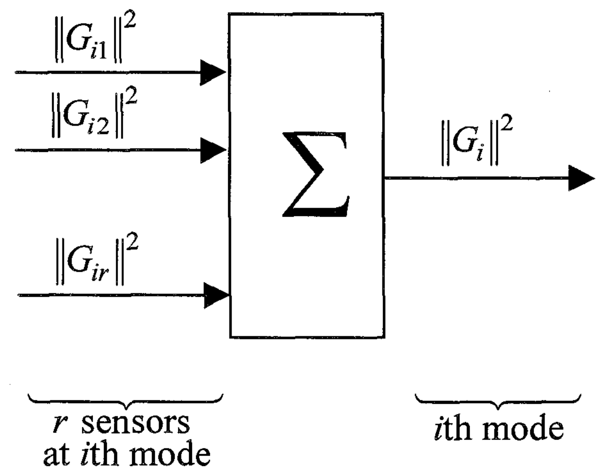
Norms of a mode with sensors

The H_2, H_∞, and Hankel norms of the ith mode of a structure with a set of r sensors is the rms sum of norms of the mode with each single sensor from this set, i.e.,

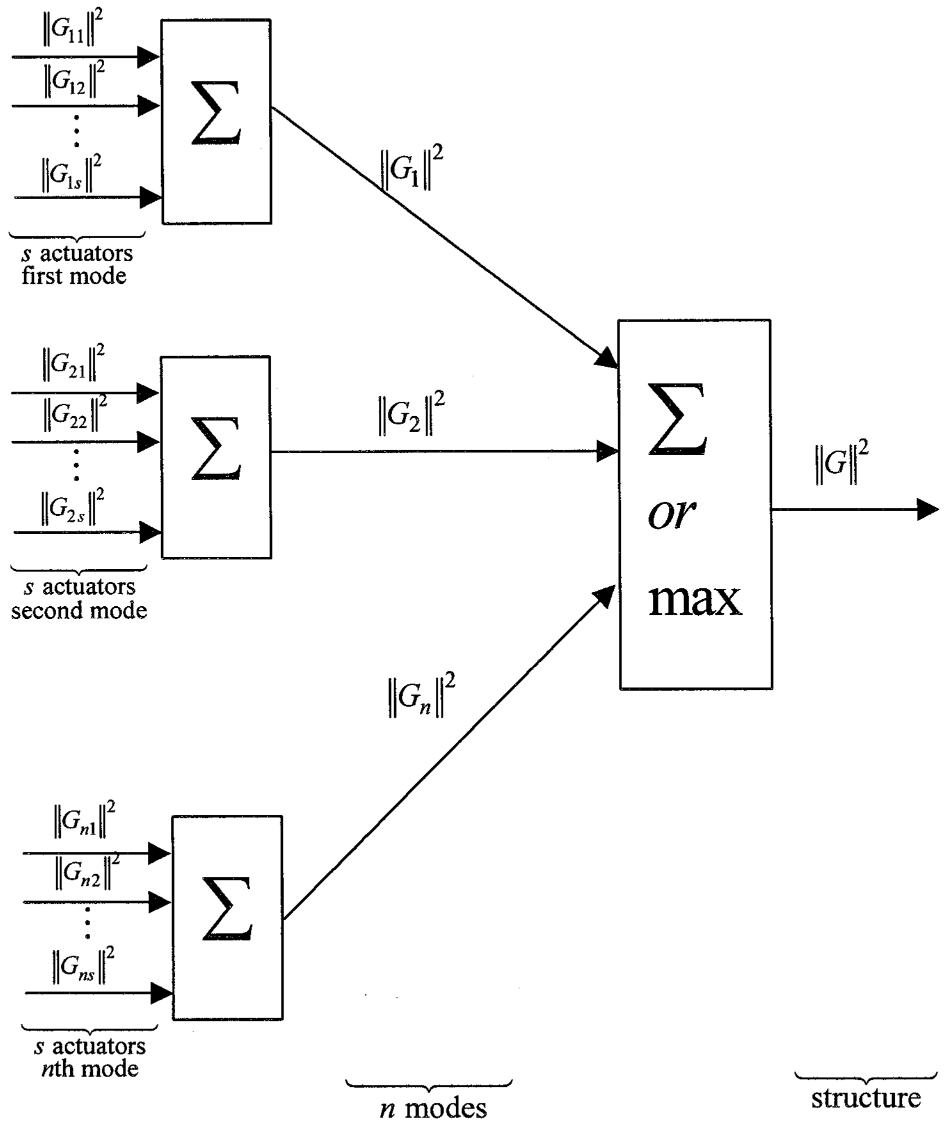
$$\|G_i\| \cong \sqrt{\sum_{j=1}^{r} \|G_{ij}\|^2}$$

Norms of a mode with a/s





Decomposition of norms of a structure



Using H_2 norm to structural damage detection

The jth sensor norm of a healthy structure $\left\| G_{shj} \right\|_2$

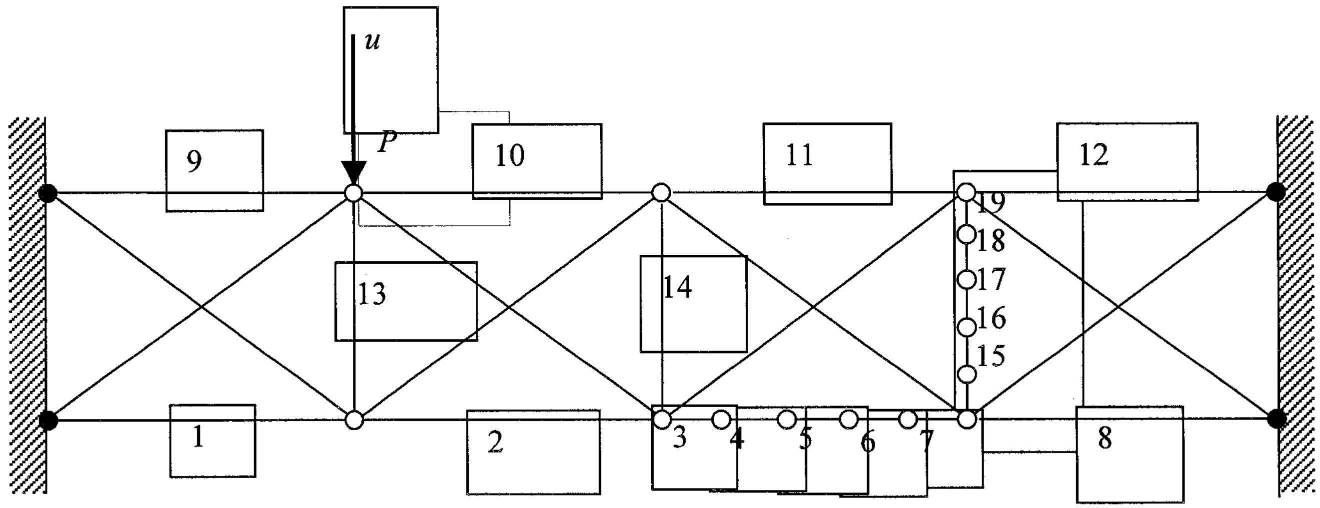
and the jth sensor norm of a damaged structure $\left\| G_{sdj} \right\|_2$

The jth sensor index of the structural damage is a weighted difference between the jth sensor norm of a healthy and damaged structure, i.e.,

$$\boxed{\sigma_{sj} = \frac{\left| \left\| G_{shj} \right\|_2^2 - \left\| G_{sdj} \right\|_2^2 \right|}{\left\| G_{shj} \right\|_2^2}}$$

The index reflects impact of the structural damage on the ith mode.

Damage detection (cont.)



The cross section area of the steel beams is of 1 cm^2.

Two damage cases are considered:

1. a 20% reduction of the stiffness of the beam No 5
2. a 20% reduction of the stiffness of the beam No17.

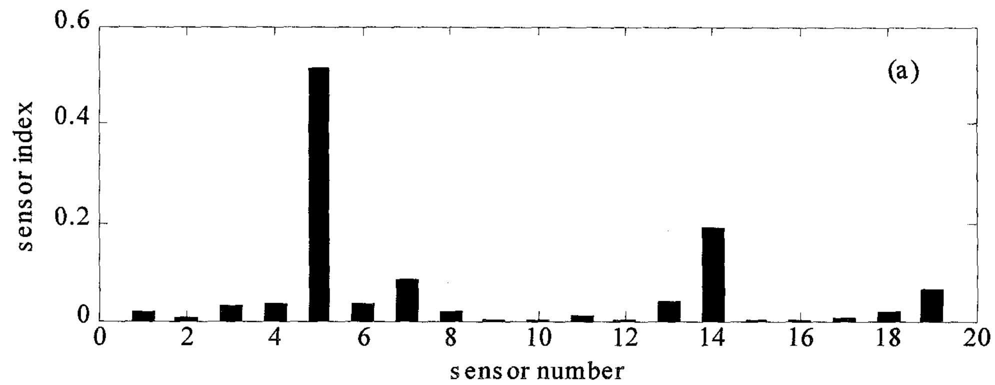
   The structure was more densely divided near the damage locations to better reflect the stress concentration.

   Nineteen strain-gage sensors are placed at the beams 1 to 19.

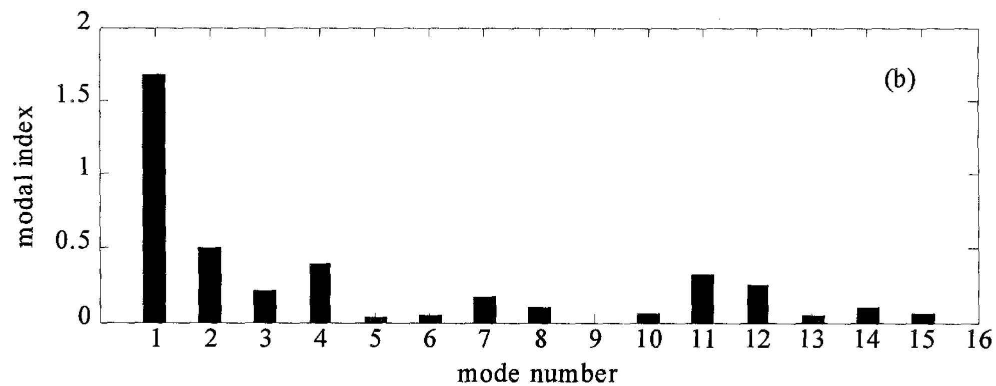
   A vertical force applied at node *P* excites the structure.

Damage detection (first case)

Sensor indices



Modal indices



The sensor indices in (a) indicate that the sensor No 5, suffered the most changes.
The modal indices in (b) show that the first mode was heavily affected by the damage.

Damage detection (second case)

Sensor indices

Modal indices

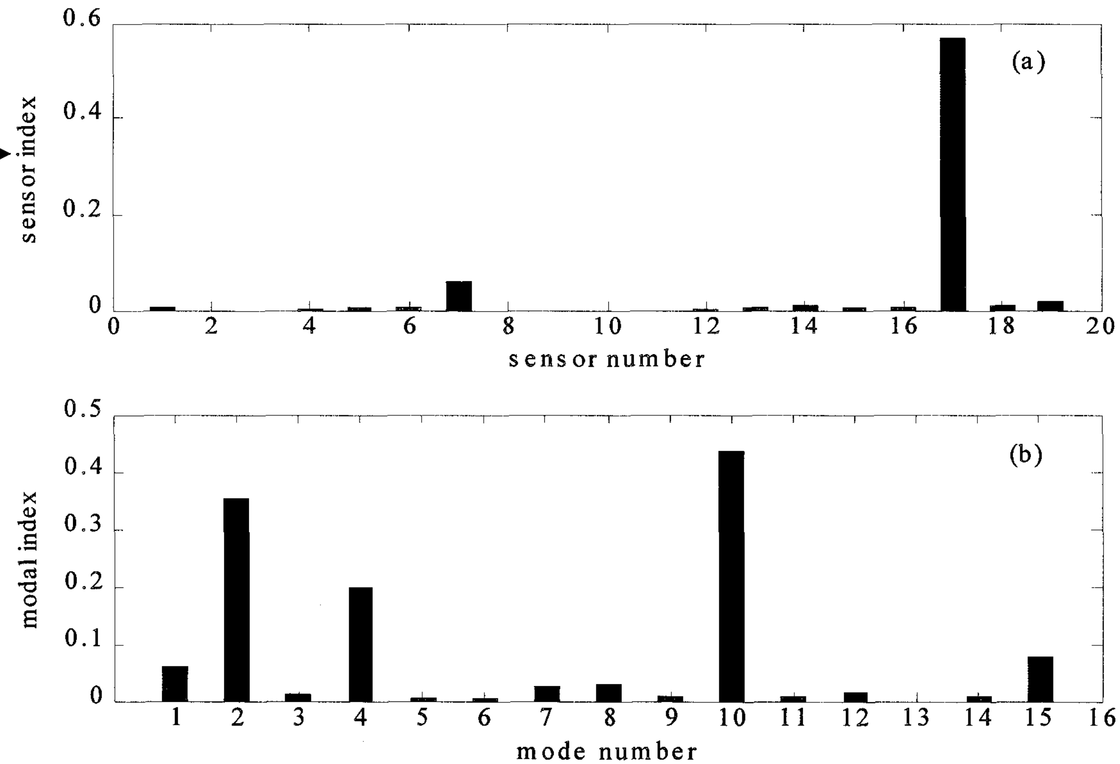
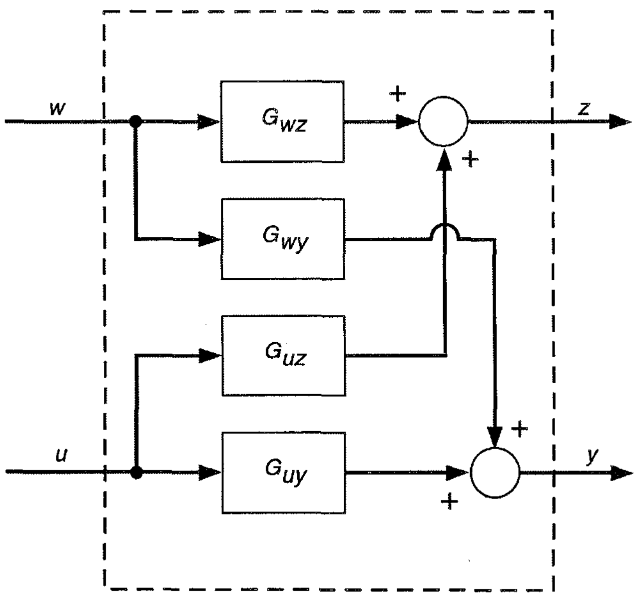


Fig.(a) shows the largest sensor index at location No 17.
The modal indices in Fig.(b) show that the tenth and the second modes were mostly affected by the damage.

Norms of a structure in a general configuration

The following norm relationships holds:



$$\left\|G_{wzi}\right\|\left\|G_{uyi}\right\| \cong \left\|G_{wyi}\right\|\left\|G_{uzi}\right\|$$

Norms of a structure in a general configuration

This property is important for the closed-loop design. For the plant one obtains

$$z = G_{wz} w + G_{uz} u \qquad\qquad y = G_{wy} w + G_{uy} u$$

The closed-loop transfer matrix from w to z, with the controller K such that $u = Ky$, is as follows:

$$G_{cl} = G_{wz} + G_{uz} K (I - G_{uy} K)^{-1} G_{wy}$$

From the above equation it follows that the controller impacts the closed-loop performance not only through the action from u to y, but also through the cross-actions from u to z, and from w to y.

Norms of a structure in a general configuration

Therefore, if

G_{wy} G_{uz} are zero, the controller has no impact whatsoever on the performance *z*. Thus the controller design task consists of simultaneous gain improvement between *u* and *y*, *w* and *y*, and *u* and *z*.

However, the above property shows that the improvement in G_{uy} automatically leads to the improvement of G_{wy} and G_{uz}.

Thus, the task of actuator and sensor location simplifies to the manipulation of G_{uy} alone

Part 3
Actuator and sensor placement

The placement problem:

☛ given an initial (large) set of sensors and actuators,

☛ determine the locations of a smaller subset of sensors or actuators

☛ such that the H_2 , H_∞, or Hankel norms of the subset is as close as possible to the norm of the original set.

H_2 Placement Indices and Matrices

Placement index:

$$\sigma_{2ki} = w_{ki} \frac{\|G_{ki}\|_2}{\|G\|_2}$$

Placement matrix:

$$\begin{array}{cc} \Sigma_2 = \begin{bmatrix} \sigma_{211} & \sigma_{212} & \cdots & \sigma_{21k} & \cdots & \sigma_{21S} \\ \sigma_{221} & \sigma_{222} & \cdots & \sigma_{22k} & \cdots & \sigma_{22S} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \sigma_{2i1} & \sigma_{2i2} & \cdots & \sigma_{2ik} & \cdots & \sigma_{2iS} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \sigma_{2n1} & \sigma_{2n2} & \cdots & \sigma_{2nk} & \cdots & \sigma_{2nS} \end{bmatrix} & \begin{array}{c} \\ \\ \\ \Leftarrow i\text{th mode} \\ \\ \\ \end{array} \\ \Uparrow & \\ k\text{th actuator} & \end{array}$$

Actuator/Sensor Indices

H_2 actuator/sensor index is the rms sum of the *k*th actuator/sensor indexes over all modes

$$\sigma_{ak} = \sqrt{\sum_{i=1}^{n} \sigma_{ik}^2}$$

For the H_∞ and Hankel norms it is the largest index over all modes

$$\sigma_{ak} = \max_i (\sigma_{ik})$$

A/S and modal indices

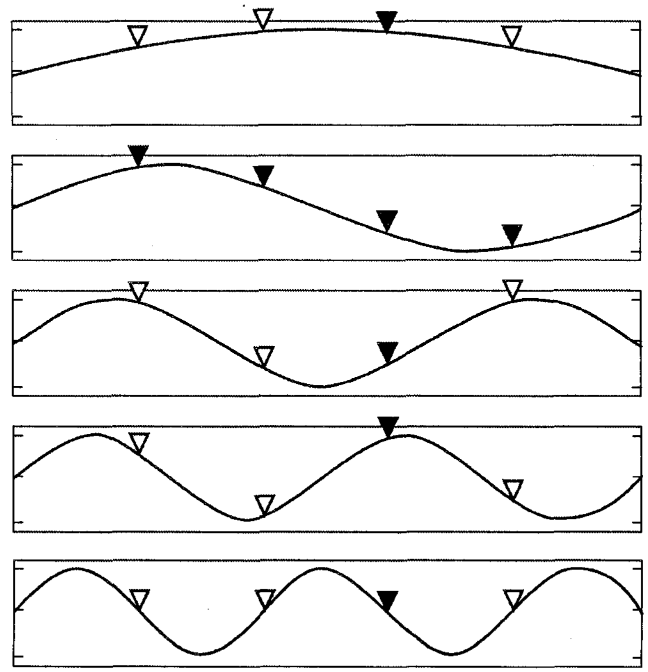
Determination of the H_2 actuator and modal indices of a pinned beam

∇– actuator location;

$\blacktriangledown$ – actuators used for the calculation of the indices.

$$\sigma_{a4} = \sqrt{\sigma_{14}^2 + \sigma_{24}^2 + \sigma_{34}^2 + \sigma_{44}^2 + \sigma_{54}^2} \Downarrow$$

$$\sigma_{m2} = \sqrt{\sigma_{21}^2 + \sigma_{22}^2 + \sigma_{23}^2 + \sigma_{24}^2} \Rightarrow$$



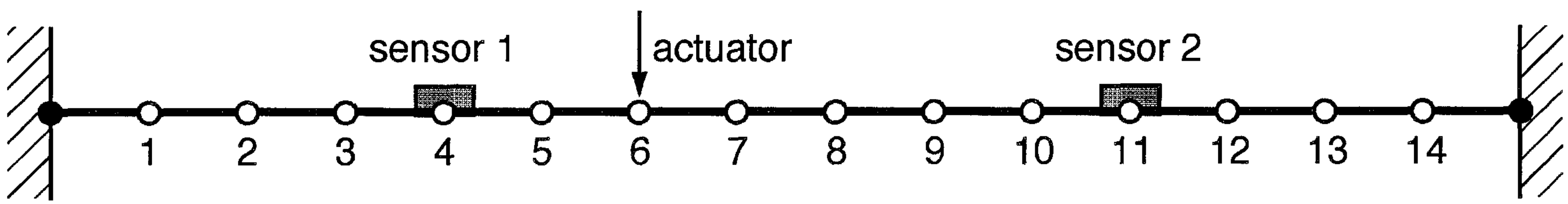
Modal indices

The *i*th mode index is a rms sum of the indices over all actuators/sensors

$$\sigma_{mi} = \sqrt{\sum_{k=1}^{S} \sigma_{ik}^2}$$

Example

Placing two sensors on a beam for best sensing of up to 4 modes



A beam with one actuator and two sensors

Using the H_∞ norm find the best place for two displacement sensors in vertical direction:

- to sense the first, second, third, and fourth mode, and
- to sense simultaneously the first two modes, the first three modes, and the first four modes.

Example (cont.)

Each node of a beam has 3 degrees of freedom :
horizontal displacement x, vertical displacement y, and rotation in the figure plane θ. Denote a unit vector

$$e_i = [0, 0, \ldots, 1, \ldots, 0]$$

that has all zeros except 1 at ith location, then the displacement output matrix for sensors located at ith and jth node is

$$C_{qij} = \begin{bmatrix} e_{3i-1} \\ e_{3j-1} \end{bmatrix}$$

The input matrix is $$B_o = e_{17}^T$$

Example (cont.)

The H_∞ norm for the *k*th mode (*k*=1,2,3,4) and (*i,j*) sensor location

$$\left\|G_{ij}\right\|_\infty = \frac{\left\|B_{ij}\right\|_2 \left\|C_i\right\|_2}{2\zeta_i\omega_i}, \qquad \left\|G_{ik}\right\|_\infty = \frac{\left\|B_i\right\|_2 \left\|C_{ki}\right\|_2}{2\zeta_i\omega_i}$$

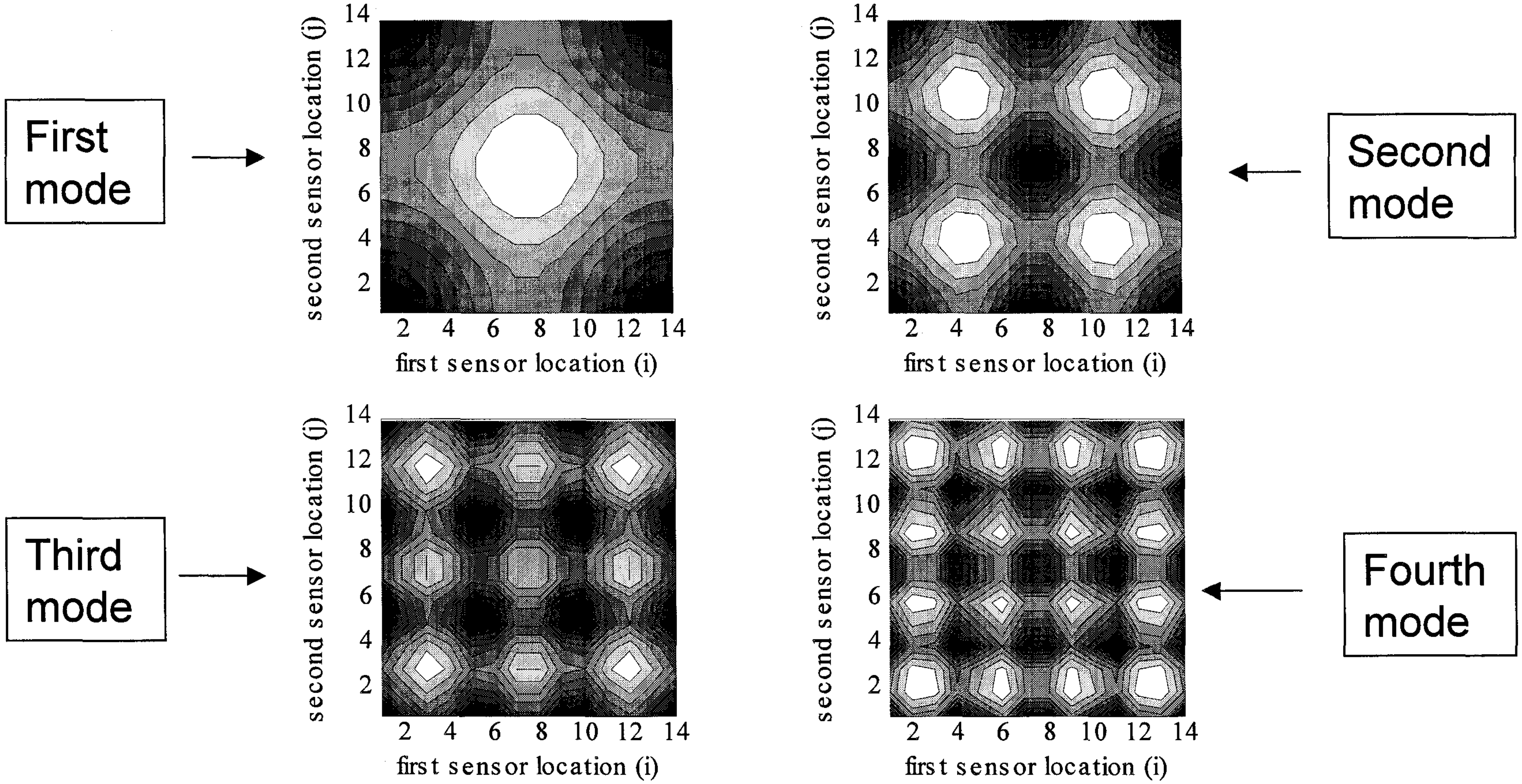
From these norms the sensor placement indices for each mode are

$$\sigma_{\infty ki} = w_{ki} \frac{\left\|G_{ki}\right\|_\infty}{\left\|G\right\|_\infty}$$

using weight such that

$$\max_{i,j}(\sigma_{\infty kij}) = 1$$

Example (cont.)



Sensor placement indices as a function of sensor locations

Example (cont.)

Next, the indices for the first two modes are determined

$$\sigma_{\infty 12ij} = \max(\sigma_{\infty 1ij}, \sigma_{\infty 2ij})$$

the indices for the first three modes

$$\sigma_{\infty 123ij} = \max(\sigma_{\infty 1ij}, \sigma_{\infty 2ij}, \sigma_{\infty 3ij})$$

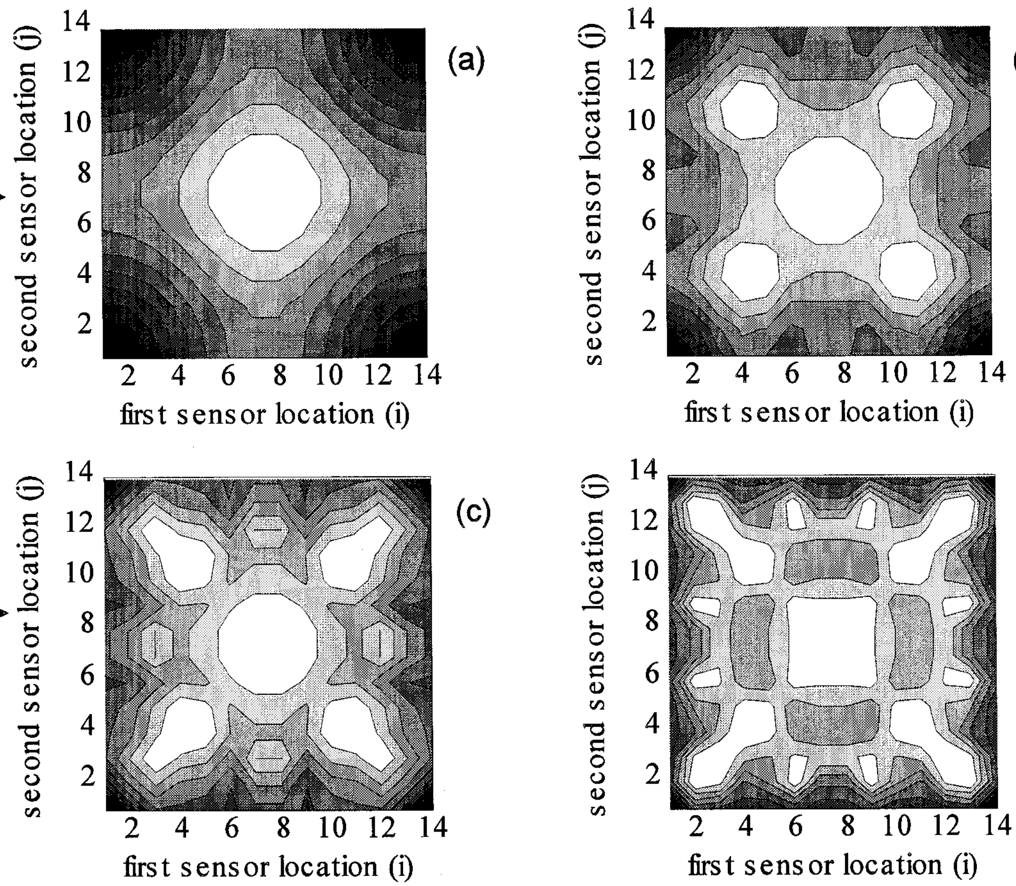
the indices for the first four modes

$$\sigma_{\infty 1234ij} = \max(\sigma_{\infty 1ij}, \sigma_{\infty 2ij}, \sigma_{\infty 3ij}, \sigma_{\infty 4ij})$$

Example (cont.)

First mode

First two modes



First three modes

First four modes

Sensor placement H_∞ indices as a function of sensor locations

Example (cont.)

Two sensor placement using the H_2 norms. First, the H_2 norm for the *k*th mode (*k*=1,2,3,4) and (*i,j*) sensor location is obtained

$$\left\|G_{ij}\right\|_2 = \frac{\left\|B_{ij}\right\|_2 \left\|C_i\right\|_2}{2\sqrt{\zeta_i \omega_i}}, \qquad \left\|G_{ik}\right\|_2 = \frac{\left\|B_i\right\|_2 \left\|C_{ki}\right\|_2}{2\sqrt{\zeta_i \omega_i}}$$

Example (cont.)

The indices for the first two modes are determined

$$\sigma_{2,12ij} = \sqrt{\sigma_{2,1ij}^2 + \sigma_{2,2ij}^2}$$

for the first three modes

$$\sigma_{2,123ij} = \sqrt{\sigma_{2,1ij}^2 + \sigma_{2,2ij}^2 + \sigma_{2,3ij}^2}$$

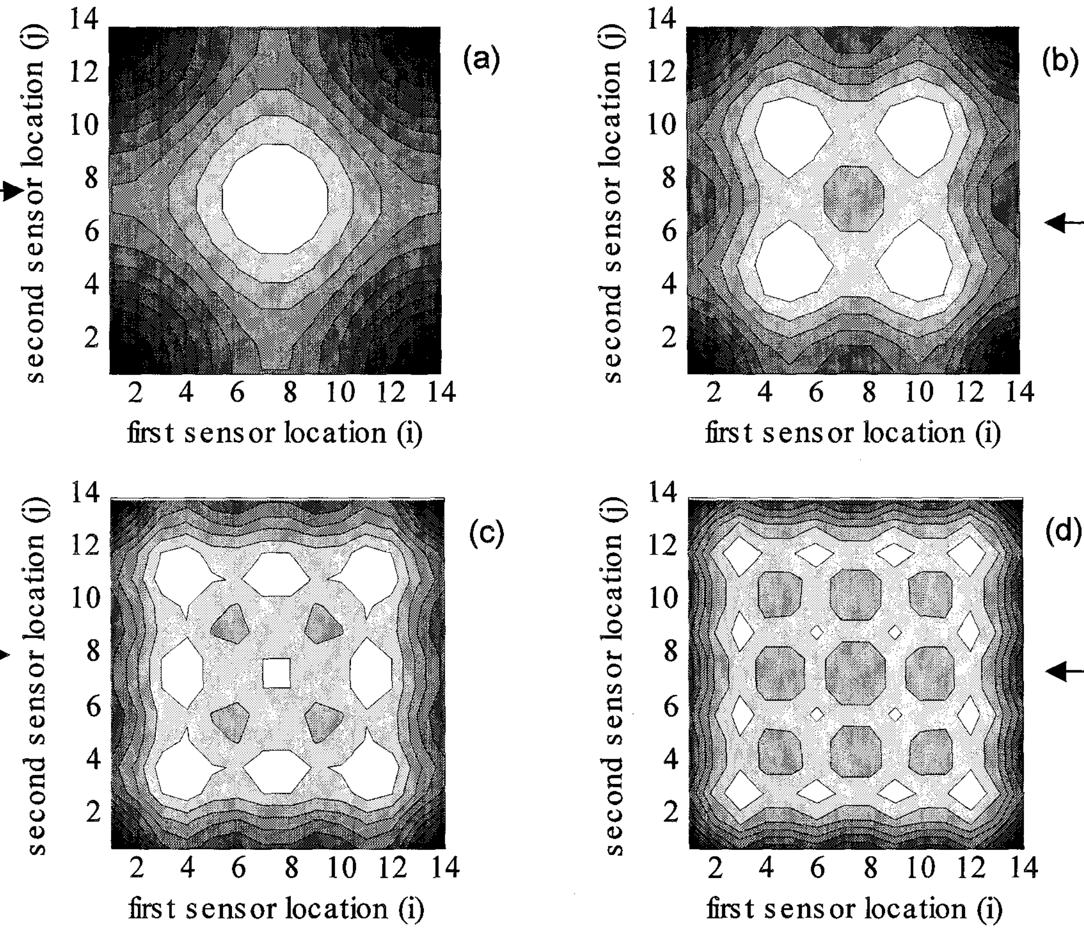
and for the first four modes

$$\sigma_{2,1234ij} = \sqrt{\sigma_{2,1ij}^2 + \sigma_{2,2ij}^2 + \sigma_{2,3ij}^2 + \sigma_{2,4ij}^2}$$

Example (cont.)

First mode

First two modes



First three modes

First four modes

Sensor placement H_2 indices as a function of sensor locations

Placement for Large Structures

For a large structure suppose that a specific sensor location gives a high-performance index. Inevitably, locations close to it will have a high-performance index as well. But the locations in the neighborhood of the original sensor are not necessary. Additional criterion will eliminate closely spaced sensors/actuators. Define a vector of the *i*th sensor norms, which is composed of the squares of the modal norms

$$g_i = \begin{bmatrix} \|G_{i1}\|^2 \\ \|G_{i2}\|^2 \\ \vdots \\ \|G_{in}\|^2 \end{bmatrix}$$

G_{ik} denotes the transfer function of the *k*th mode at the *i*th sensor

Placement for Large Structures

The correlation coefficient

$$r_{ik} = \frac{g_i^T g_k}{\|g_i\|_2 \|g_k\|_2}$$

The membership index $I(k)$,

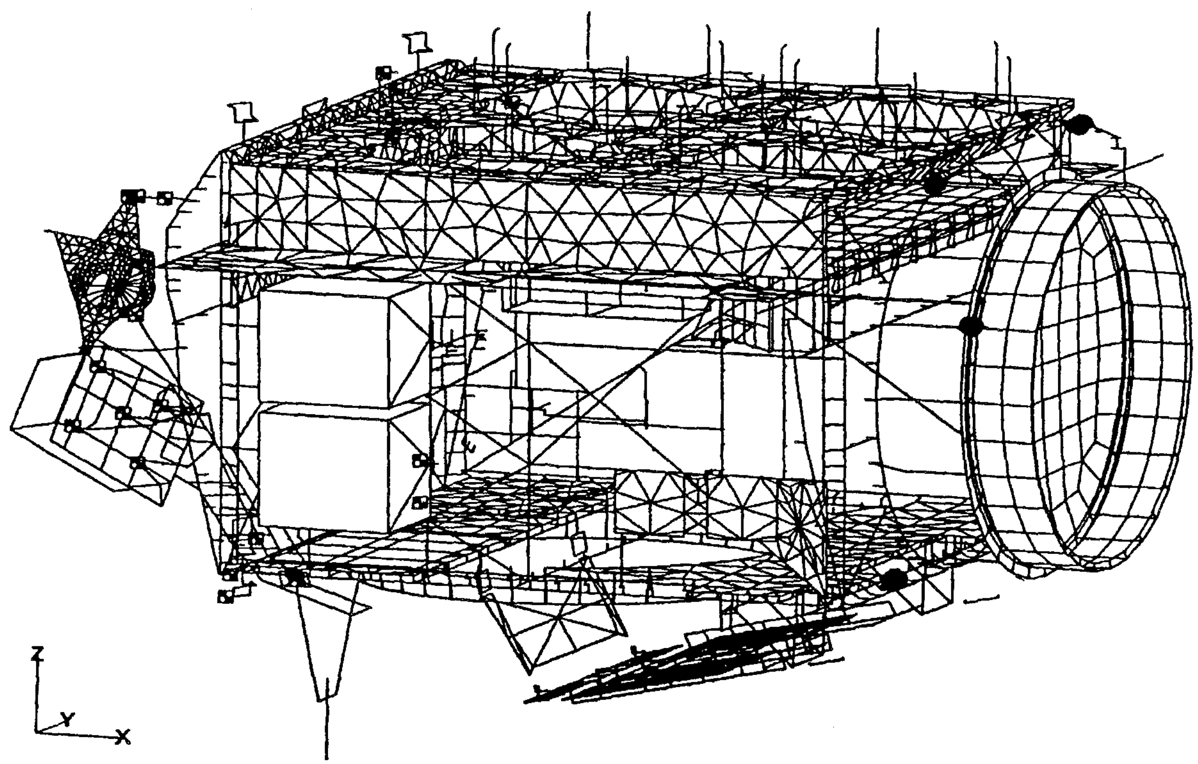
$$\boxed{I(k) = \begin{cases} 0 & \text{if } r_{ik} > 1-\varepsilon \quad \text{and} \quad \sigma_k \le \sigma_i \quad \text{for } k > i \ , \\ 1 & \text{elsewhere} \ , \end{cases}}$$

$$\varepsilon = 0.01 - 0.20$$

If $I(k)$=1, the kth sensor is accepted, and if $I(k)$=0, the kth sensor is rejected (in this case the two locations i and k are either highly correlated, or the ith location has a higher performance).

ISS example

The Z1 module of the International Space Station structure



ISS example (cont.)

The finite-element model of the structure consists of 11,804 degrees of freedom with 56 modes below the frequency of 70 Hz. The task is to identify all modes below 70 Hz in tests, with accelerometers used as sensors .

Actuator Placement. The selection of four actuator locations.

The initial selection procedure combines engineering judgment, practical experience, and physical constraints including the following criteria:

- All target modes should be excited with relatively equal amplitudes.
- The structure is excited in three axes.

The structure drawings and the finite-element model were examined and 2256 actuator candidate locations were selected out of the 11,804 translational degrees of freedom. The selection was based on accessibility of the locations, strength of the structural parts, modal masses, and local flexibility.

ISS example (cont.)

- The Hankel norms of each actuator were determined and used to evaluate the actuator importance indices. For each of 56 modes the six most important actuators were selected, obtaining 268 actuator locations. Next, the correlation coefficients of the Hankel norm vectors for each actuator location were obtained. Those highly correlated were discarded. The number of actuators was reduced down to 52 locations.

- The next step involved evaluation of the location of the actuators using the finite-element model simulations, along with determination of accessibility, structural strength, and the importance index. The final four actuators were located at the nodal points shown in the above figure as black spots. These four locations are essentially near the four corners of the structure.

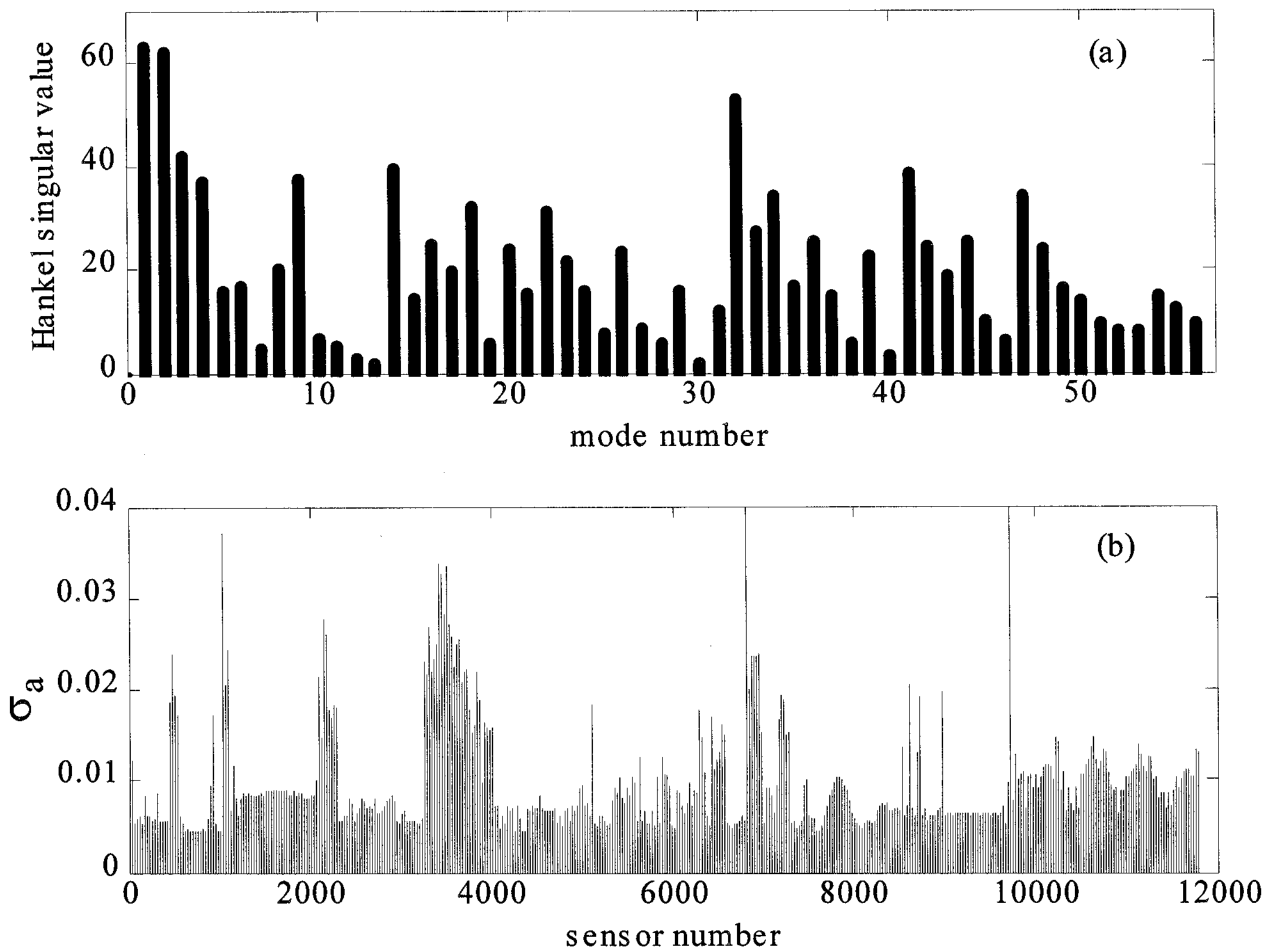
ISS example (cont.)

- *Sensor Placement.* The sensor selection criteria includes:
  - • Establishing the maximum allowable number of sensors. In our case it was 400.
  - • Determination of the sensor placement indices for each mode. Sensors with the highest indices were selected.
  - • Using the correlation procedure to select uncorrelated sensors by evaluating the membership index.

- The excitation level of each mode by the four selected actuators is represented by the Hankel norms, the next figure. It can be seen that some modes are weakly excited, providing weaker measurement signal, thus they are more difficult to identify. The next figure also presents an overview of the sensor importance index for each sensor.

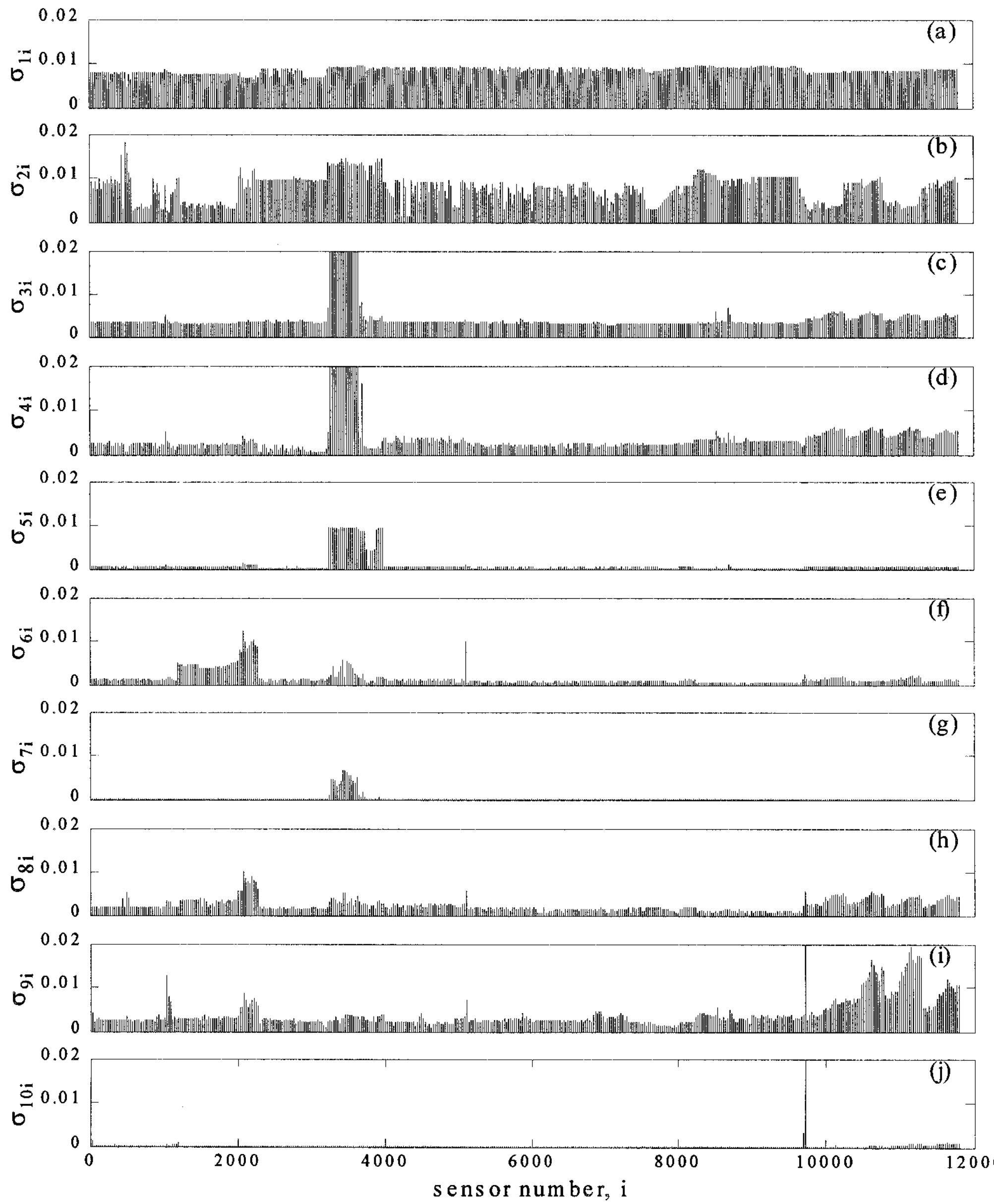
ISS example (cont.)

(a) HSV, and (b) sensor index for all modes



ISS example (cont.)

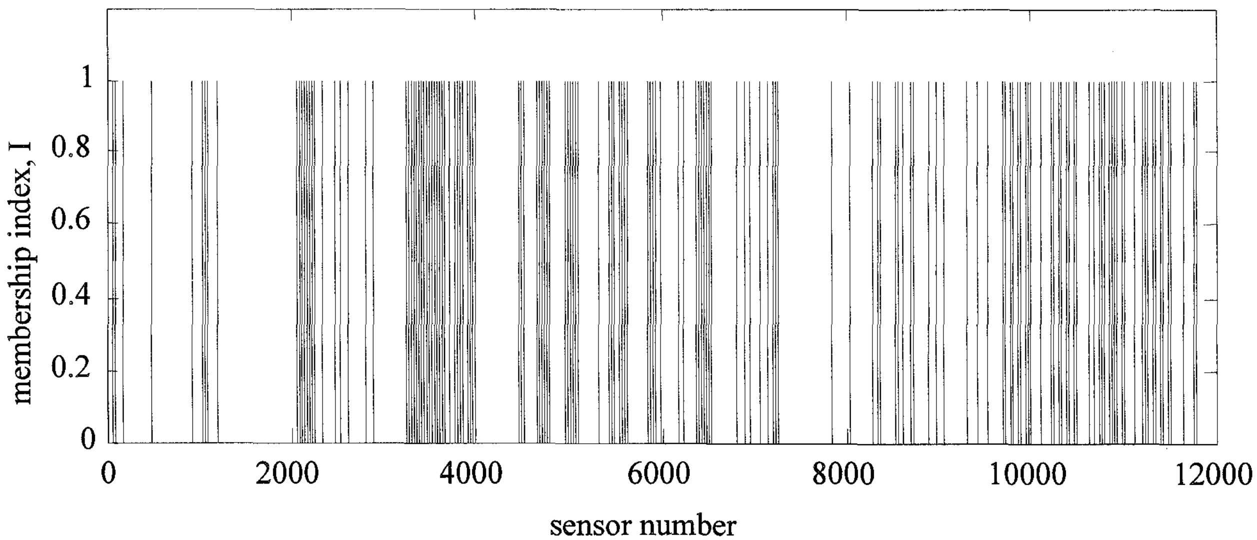
- The set of illustrations presented in the next figure shows the placement indices of each sensor for the first 10 modes.

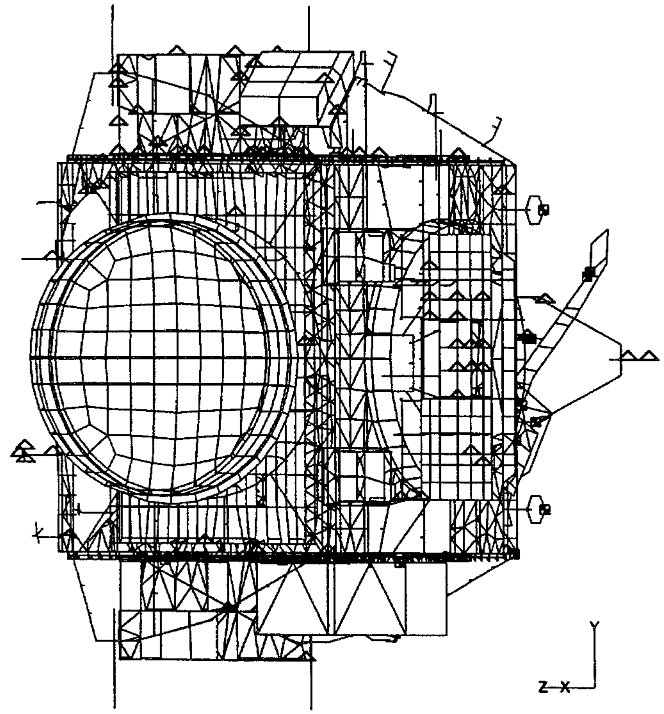


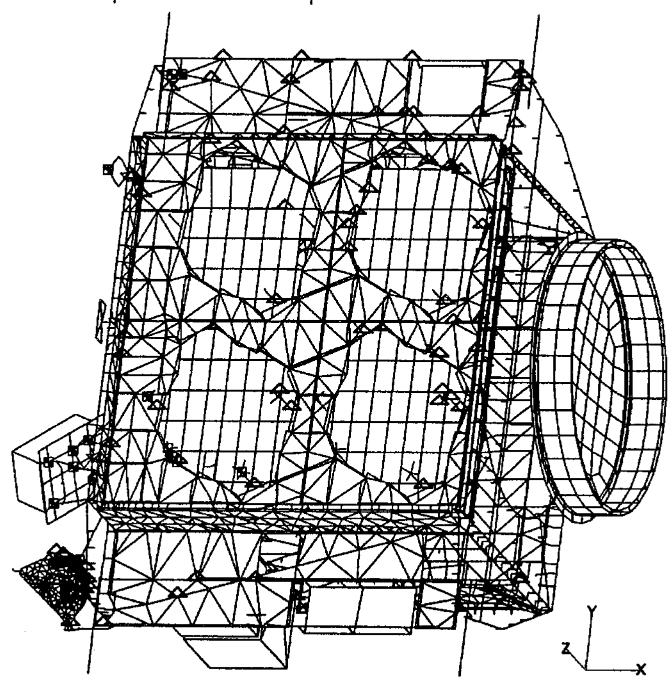
0.02
0.01
0
σ_{1i}
(a)
σ_{2i}
(b)
σ_{3i}
(c)
σ_{4i}
(d)
σ_{5i}
(e)
σ_{6i}
(f)
σ_{7i}
(g)
σ_{8i}
(h)
σ_{9i}
(i)
σ_{10i}
(j)
0
2000
4000
6000
8000
10000
12000
sensor number, i

ISS example (cont.)

- This figure shows the membership index *I* which has nonzero values for 341 locations.




Y
Z X

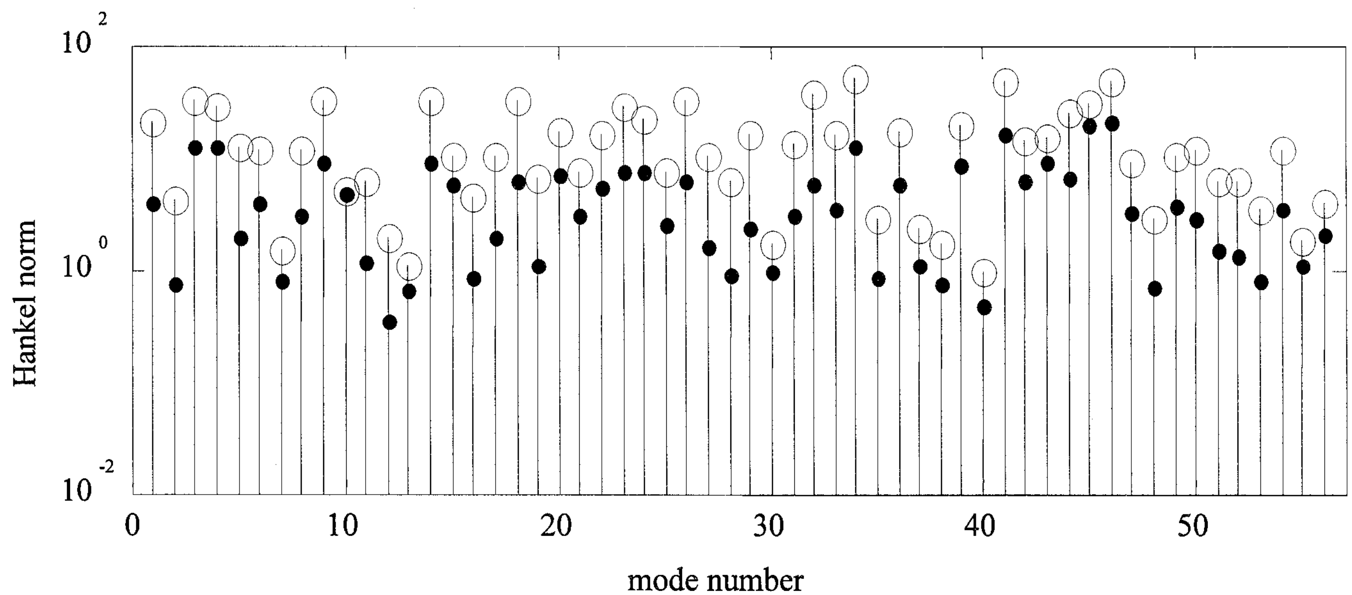

Y
Z
X

ISS example (cont.)

- Triangles indicate the selected sensor locations.
- Many of the sensors are located in and around the control moment gyros and the cable tray, since 13 out of the 56 modes involve extensive control moment gyro movement and 9 are mostly cable tray modes.
- Many of the 56 modes are local modes that require concentrations of sensors at the particular locations seen in the figure above.
- In order to test the effectiveness of the procedure we compared the Hankel norms of each mode, for the structure with a full set of 11,804 sensors, and with the selected 341 sensors. The norms with the selected sensors should be proportional to the norms of the full set (they are always smaller than the norms of the full set, but proportionality indicates that each mode is excited and sensed comparatively at the same level). The norms are shown in the next figure, showing that the profile of the modal norms is approximately preserved.

ISS example (cont.)

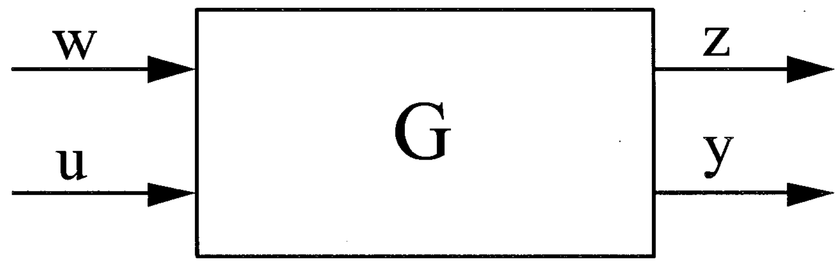
- The modal Hankel norms of the full set of sensors (○), and the selected sensors (•) of the International Space Station truss.



Placement for the General Plant Configuration

- Selection of actuators not collocated with disturbances, and sensors not collocated with the performance outputs
- The structural test plan is based on the available information on the structure itself, on disturbances acting on the structure, and on the expected structural performance. The first information is typically in a form of a structural finite-element model. The disturbance information includes disturbance location and spectral contents. The structure performance is commonly evaluated through the displacements or accelerations at selected locations.

Placement for the General Plant



- The formulation of structural testing is based on a block diagram as above. The structure input is composed of two inputs:
- the disturbances (*w*), and the actuator inputs (*u*).
- The plant output is divided into two sets:
- the performance (*z*), and the sensor output (*y*).
- The actuator inputs include forces and torque applied during a test. The disturbance inputs include disturbances, noises, and commands. The sensor signals consist of structure outputs recorded during the test. The performance output includes signals that characterize the system performance, and is not generally measured during the test.

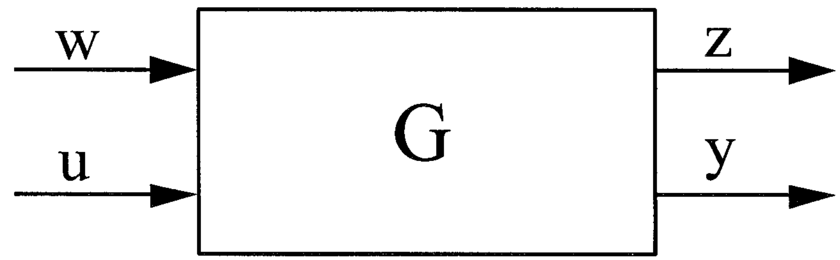
Placement for the General Plant

- to obtain the performance of the test item close to the performance of a structure in a real environment one uses the available (or candidate) locations of actuators and sensors and formulates the selection criteria to imitate the actual environment as close as possible.

Placement for the General Plant

- Modal Norms of a Generalized Structure

$$\left\|G_{wzi}\right\|\left\|G_{uyi}\right\| \cong \left\|G_{wyi}\right\|\left\|G_{uzi}\right\|$$



For each mode the product of norms of the performance loop (from disturbance to performance) and the control loop (from actuators to sensors) is equal to the product of the norms of the cross-couplings: between the disturbance and sensors, and between the actuators and performance.

Consequence: increasing the actuator–sensor connectivity, one increases the cross-connectivity: actuator-to-performance, and disturbance-to-sensors.

Sensors not only respond to the actuator input, but also to disturbances, and actuators not only impact the sensors, but also the performance.

Placement for the General Plant

- Additive Property of Actuators of a Generalized Structure.

$$\left\|G_{ui}\right\|^2 \cong \alpha_{wi}^2 \sum_{k=1}^{S} \left\|G_{u_k yi}\right\|^2$$

$G_{u_k yi}$ is the transfer function of the *i*th mode from the *k*th actuator to the output *y*, and α_{wi} is the disturbance weight of the ith mode,

$$\alpha_{wi} = \sqrt{1 + \frac{\left\|G_{wzi}\right\|^2}{\left\|G_{wyi}\right\|^2}}$$

Similar property holds for sensors

Placement for the General Plant

- Placement Indices and Matrices

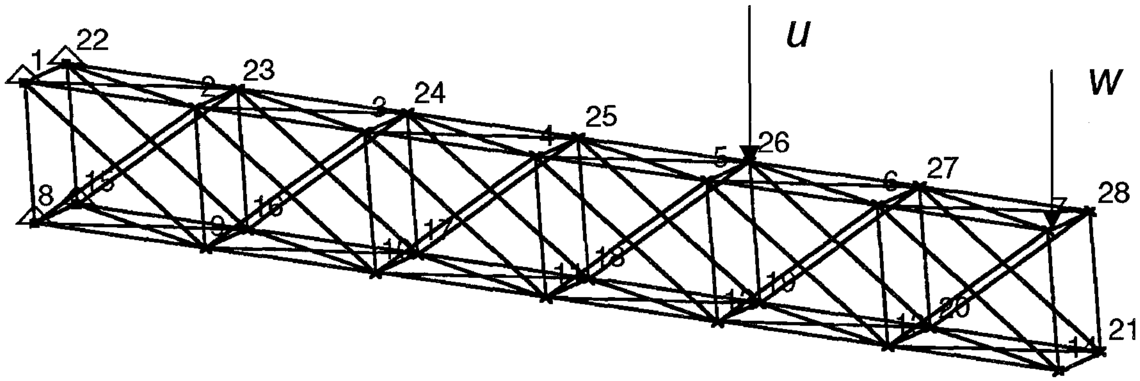
$$\sigma_{ki} = \frac{\alpha_{ui} \left\| G_{u_k yi} \right\|}{\left\| G_u \right\|}$$

$$\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1k} & \cdots & \sigma_{1S} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2k} & \cdots & \sigma_{2S} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \sigma_{i1} & \sigma_{i2} & \cdots & \sigma_{ik} & \cdots & \sigma_{iS} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_{nk} & \cdots & \sigma_{nS} \end{bmatrix} \begin{matrix} \\ \\ \\ \Leftarrow i\text{th mode} \\ \\ \\ \end{matrix}$$

$$\Uparrow$$

kth actuator

Example



3D truss

- The disturbance w is applied at node 7 in the horizontal direction.
- The performance z is measured as rates of all nodes.
- The input u is applied at node 26 in the vertical direction,
- The candidate sensor locations are at the nodes 5, 6, 7, 12, 13, 14, 19, 20, 21, 26, 27, and 28, in all three directions (total of 36 locations). Using the first 50 modes, select a minimal number of sensors that would measure, as close as possible, the disturbance-to-performance dynamics.

Example (cont.)

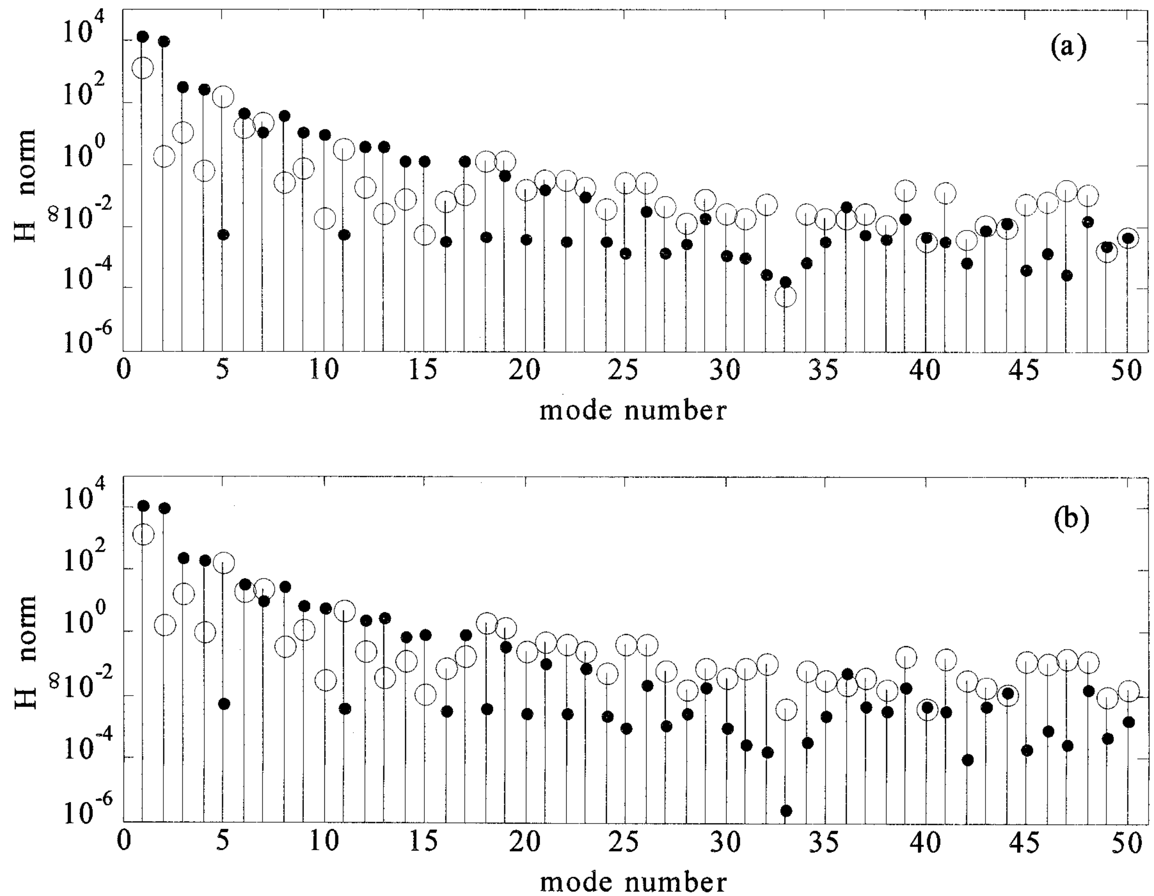
- The H_∞ norms of each mode of G_{wz} G_{uy} G_{wy} G_{uz}

G_{wz} ●

G_{uy} ○

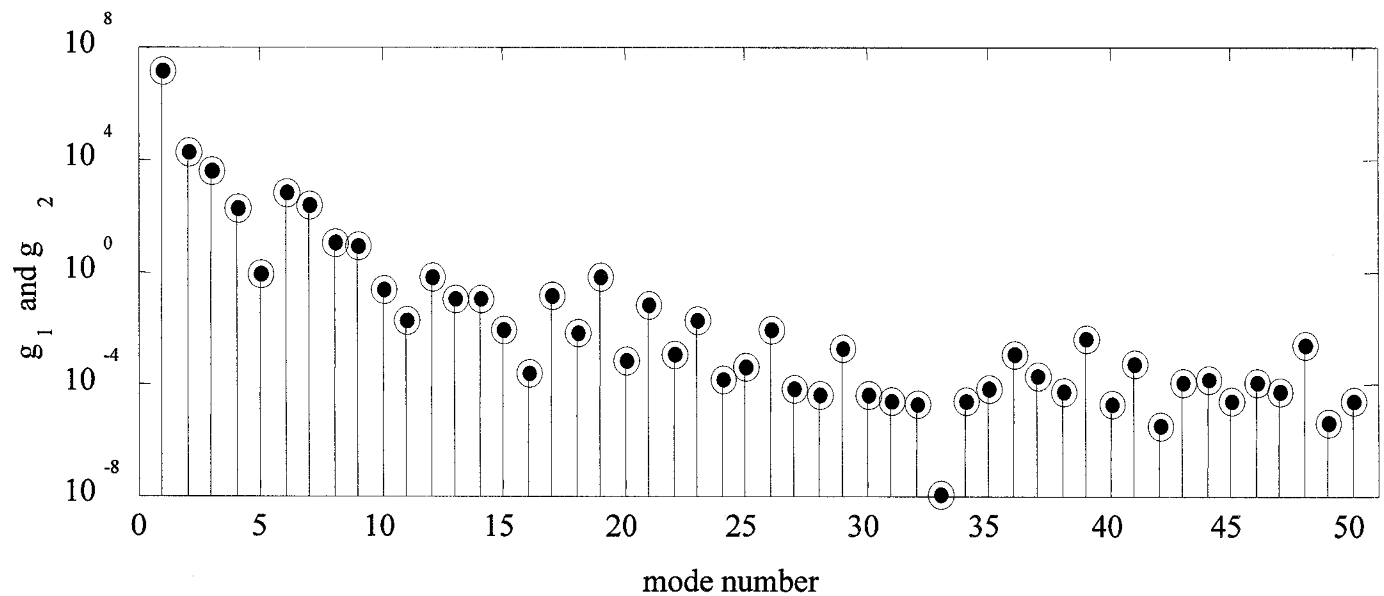
G_{wy} ●

G_{uz} ○



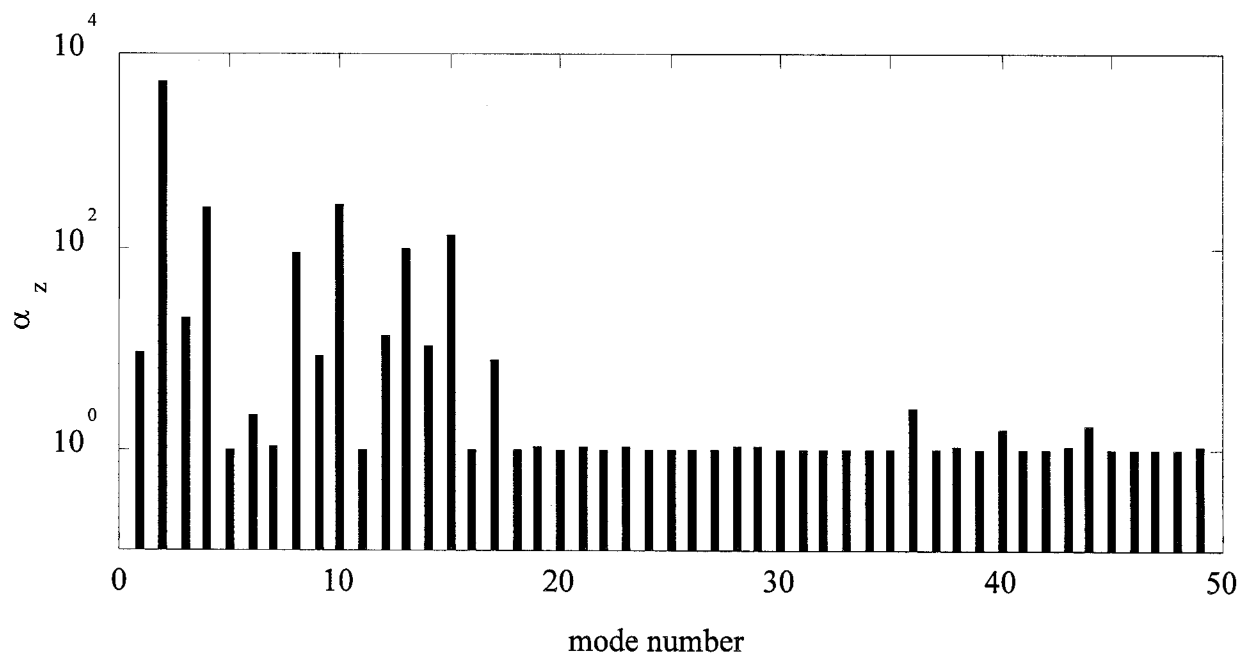
Example (cont.)

The property $\|G_{wzi}\|\|G_{uyi}\| \cong \|G_{wyi}\|\|G_{uzi}\|$ is checked. It holds since the plots of $g_1(k) = \|G_{wzk}\|_\infty \|G_{uyk}\|_\infty$ and $g_2(k) = \|G_{wyk}\|_\infty \|G_{uzk}\|_\infty$ overlap in figure below



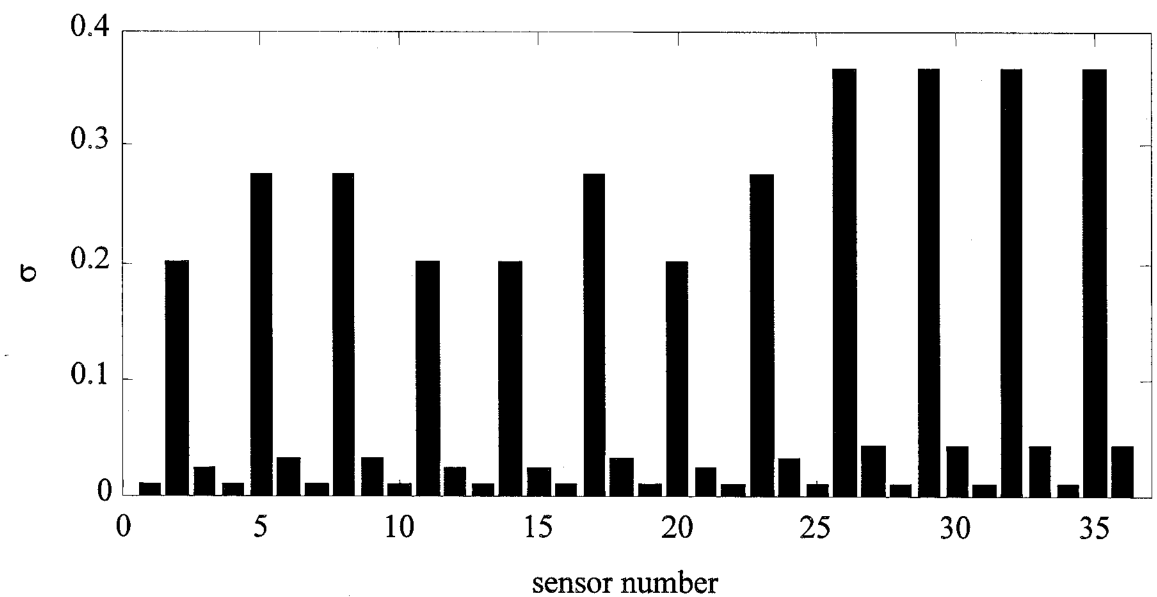
Example (cont.)

- the sensor modal weights are determined for each mode



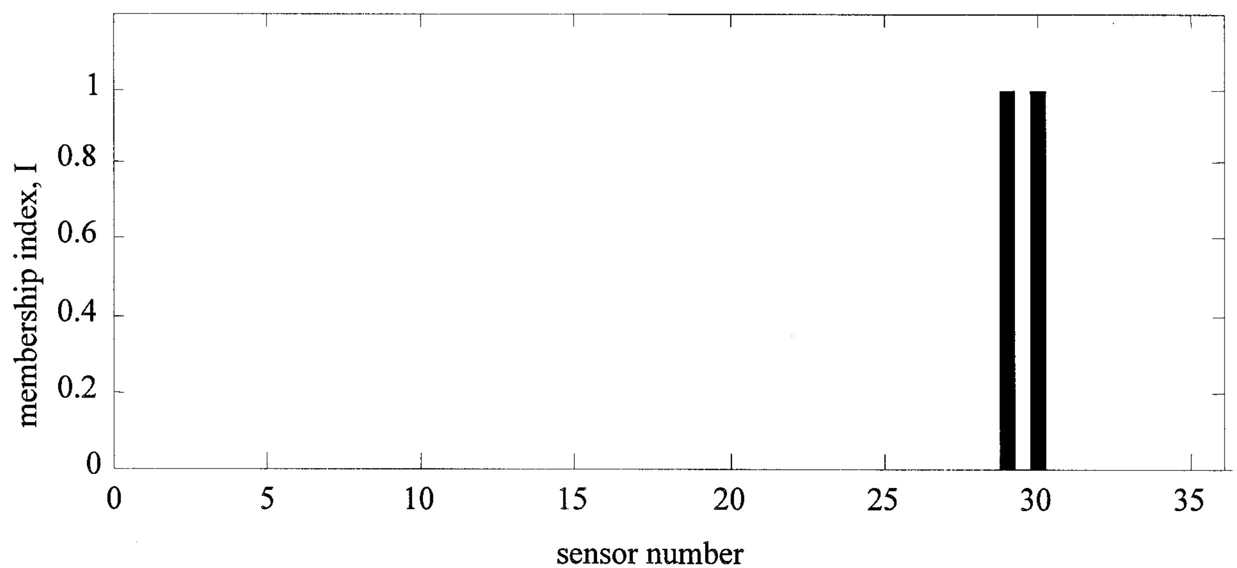
Example (cont.)

- The placement indices for each sensor are determined



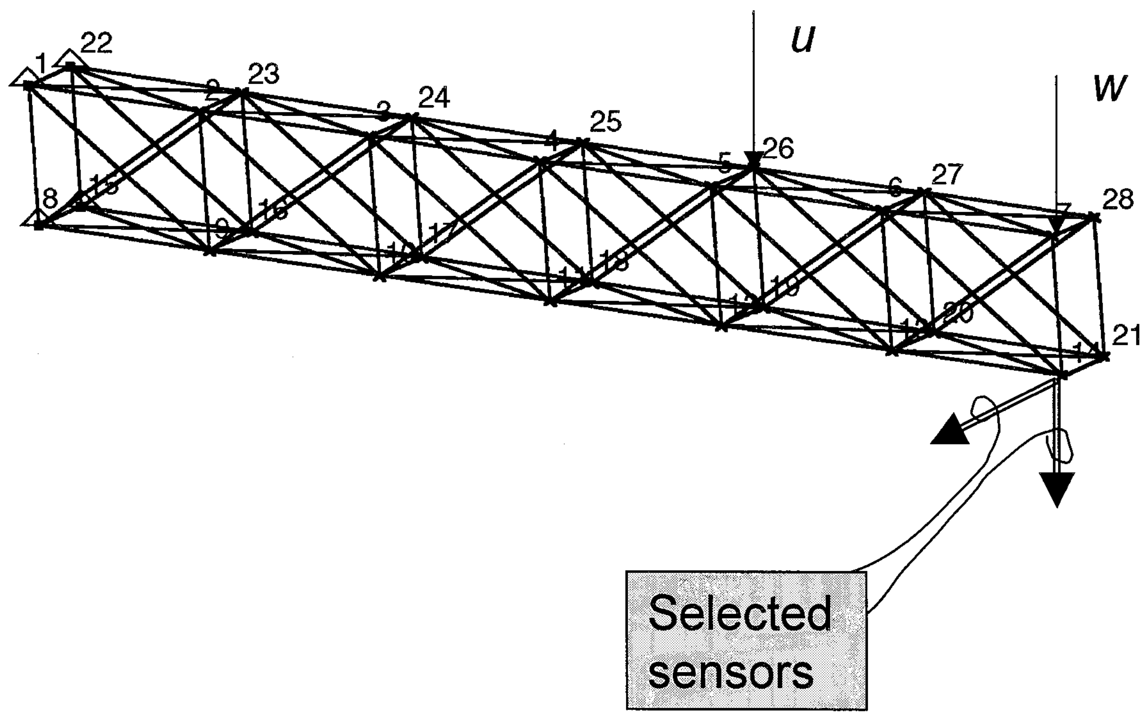
Example (cont.)

- The sensors can be highly correlated The membership index $I(k)$ is determined



The index only nonzero values are for k = 29 and k = 30, that correspond to node 14, in the y- and z-directions.

Example (cont.)



Part 4
Modal actuators and sensors

☛ In some structural tests it is desirable to isolate (i.e., excite and measure) a single mode, or selected modes.

☛We show that a proper spatial distribution of an input force are chosen to excite a single structural mode

Model used

The second order modal model

$$\ddot{q}_m + 2Z\Omega\dot{q}_m + \Omega^2 q_m = B_m u$$
$$y = C_{mq} q_m + C_{mv}\dot{q}_m$$

What happens if

$b_{mi} = 0$?

$$\ddot{q}_{mi} + 2\zeta_i\omega_i\dot{q}_{mi} + \omega_i^2 q_{mi} = b_{mi} u$$
$$y_i = c_{mqi} \quad q_{mi} + c_{mvi}\dot{q}_{mi}$$
$$y = \sum_{i=1}^{n} y_i$$

Modal actuators

Single actuator excite all modes

Excites single mode

$$B_m = \begin{Bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{Bmatrix} \qquad B_m = \begin{Bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{Bmatrix}$$

$$\ddot{q}_m + 2Z\Omega\dot{q}_m + \Omega^2 q_m = B_m u$$
$$y = C_{mq} q_m + C_{mv} \dot{q}_m$$

Each actuator excite single mode

$$B_m = \begin{Bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \end{Bmatrix}$$

Modal actuators (cont.)

- Basic equation

$$\boxed{B_m = RB_o}$$

modal

nodal

$$R = M_m^{-1}\Phi^T$$

Modal actuators (cont.)

- Modal actuator solution:

$$\boxed{B_o = R^+ B_m}$$

$$\boxed{B_o = M\Phi B_m} \longrightarrow$$

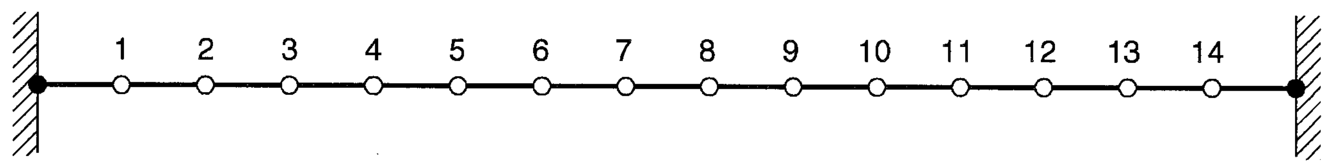
$$\Phi^T B_o = \Phi^T M \Phi B_m$$

$$\Phi^T B_o = M_m B_m$$

$$M_m^{-1} \Phi^T B_o = M_m^{-1} M_m B_m$$

$$R B_o = B_m$$

Modal actuators, example



- The vertical displacement sensors are located at nodes 2 to 15, and the single output is the sum of the sensor readings.
- Actuator locations shall be determined such that the second mode with 0.01 modal gain is excited, and the remaining modes are not excited.
- The first nine modes are considered.

Example (cont.)

- The assigned modal matrix is:

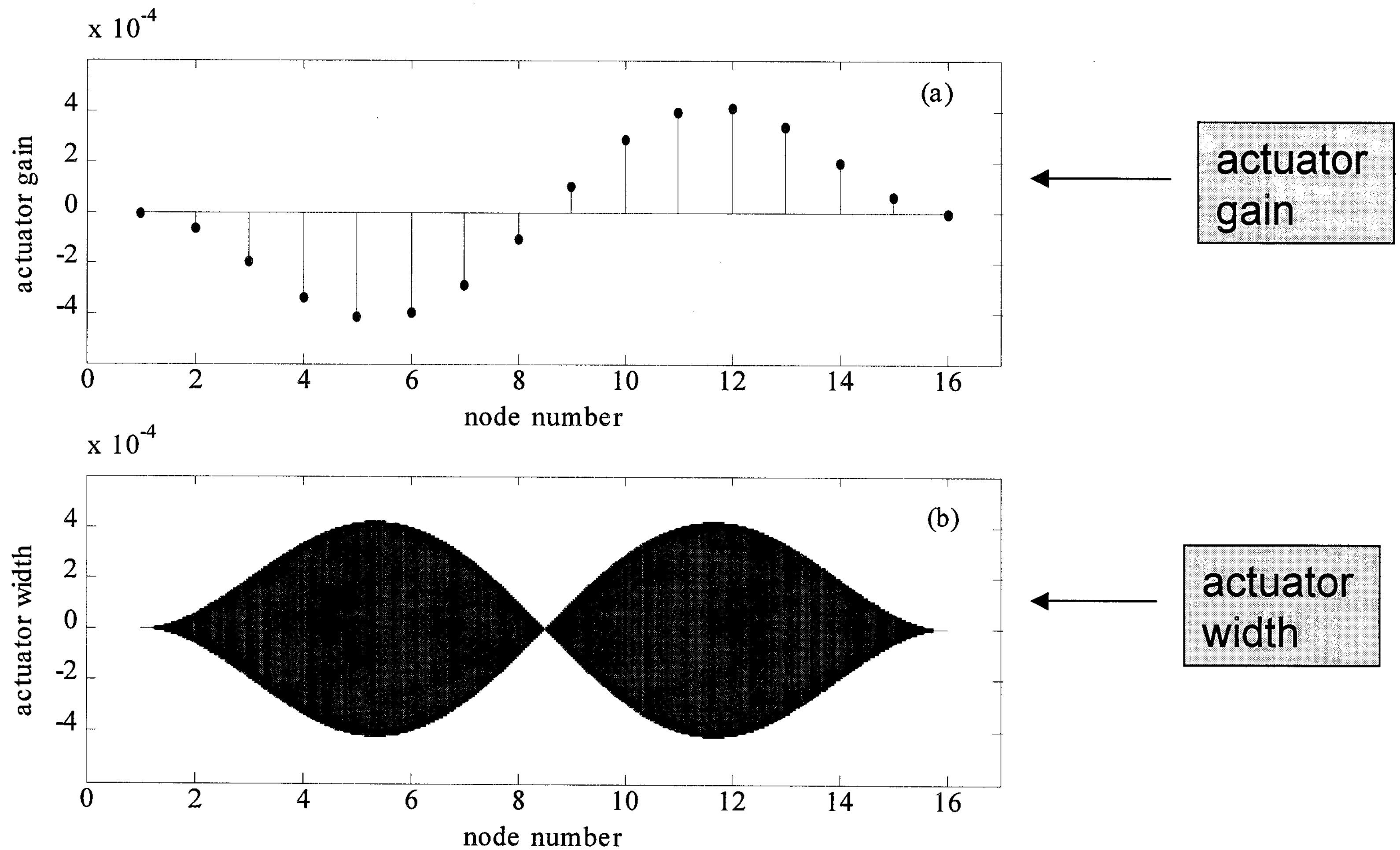
$$B_m = \begin{Bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

second mode

Example (cont.)

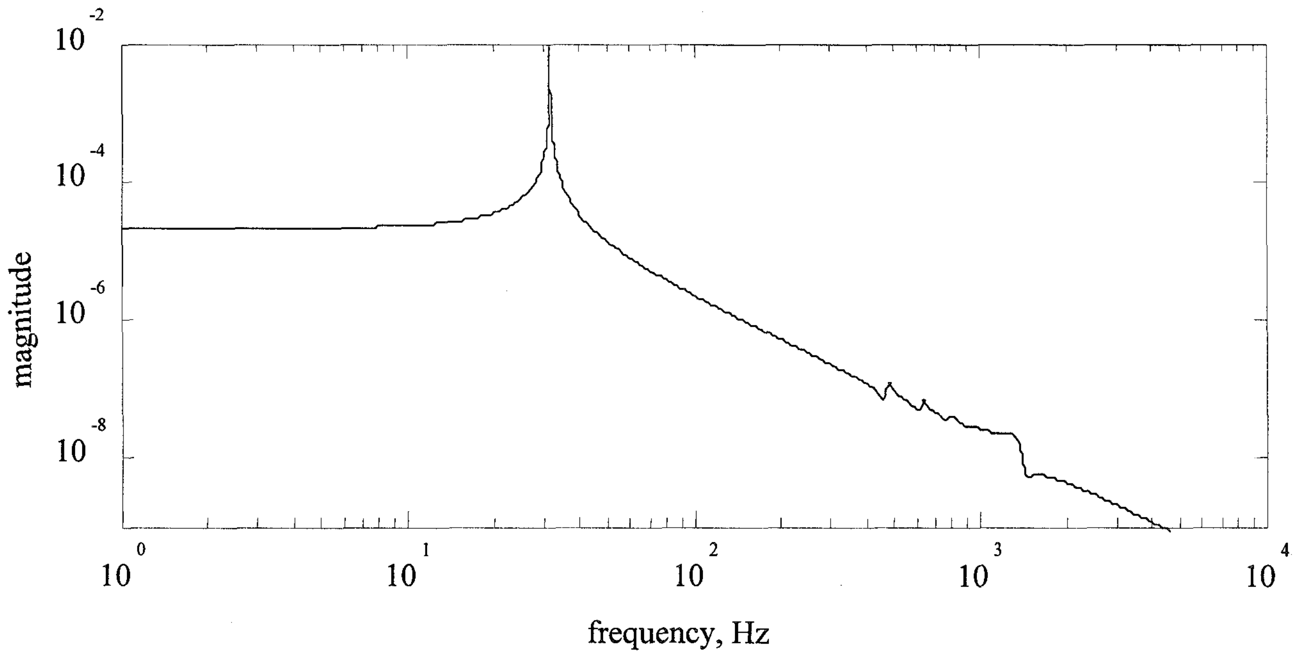
- From the modal actuator equation, for this modal input matrix, a nodal input matrix is determined.
- It contains gains for the vertical forces at the nodes 2 to 15. The gain distribution of the actuators is shown in Fig.8.1a.
- Note that this distribution is proportional to the second mode shape.

Example (cont.)



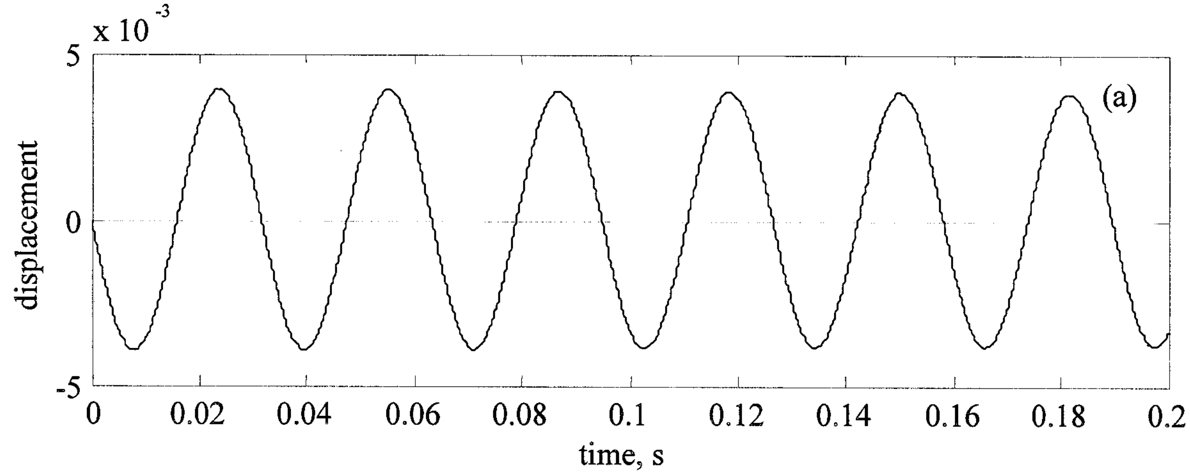
Example (cont.)

- For the input and the outputs defined as above the magnitude of the transfer function is presented in figure below. The plot shows clearly that only the second mode is excited.

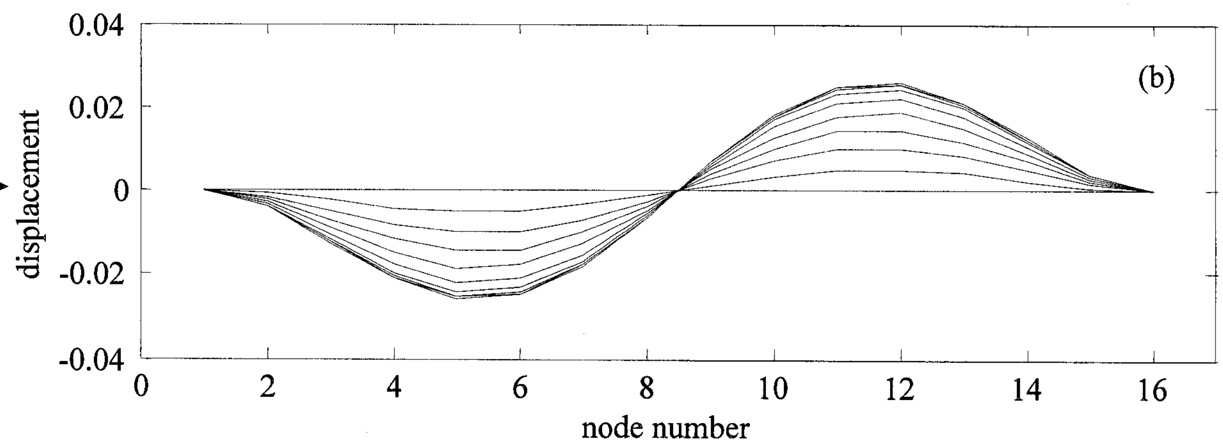


Example (cont.)

Impulse response



Nodal displace-ments for the first 10 samples



Modal actuators with assigned amplitude

- To excite the *i*th mode with amplitude a_i.
- The H_∞ norm can be used as a measure of the amplitude of the *i*th mode. In case of a single-input-single-output system the H_∞ norm of the *i*th mode is equal to the height of the ith resonance peak.

$$\|G_i\|_\infty \cong \frac{\|b_{mi}\|_2 \|c_{mi}\|_2}{2\zeta_i \omega_i}$$

Assume a unity input gain for the current mode, i.e. $\|b_{mi}\|_2 = 1$

so that the current amplitude is $$a_{oi} = \frac{\|c_{mi}\|_2}{2\zeta_i \omega_i}$$

Modal actuators with assigned amplitude

- In order to obtain amplitude a_i one has to multiply a_{oi} by the weight w_i,

$$a_i = w_i a_{oi}$$

Introducing a_{oi} as in the previous slide to the above equation one obtains

$$\boxed{w_i = \frac{2 a_i \zeta_i \omega_i}{\left\| c_{mi} \right\|_2}}$$

Define the weight matrix $W = diag(w_i)$, then the matrix that sets the required output modal amplitudes is

$$\boxed{B_{mw} = W B_m}$$

Example 2

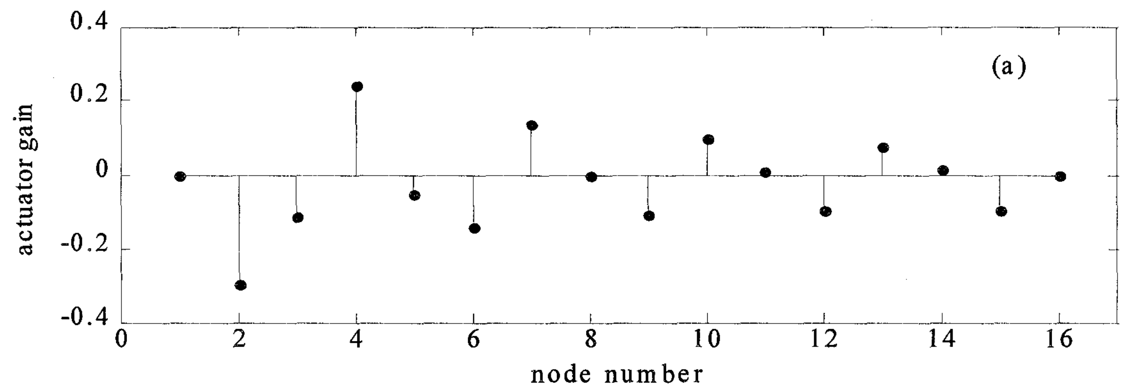
The same beam is considered. All nine modes need to be excited by a single actuator with the amplitude of 0.01. Therefore, the assigned modal input matrix is

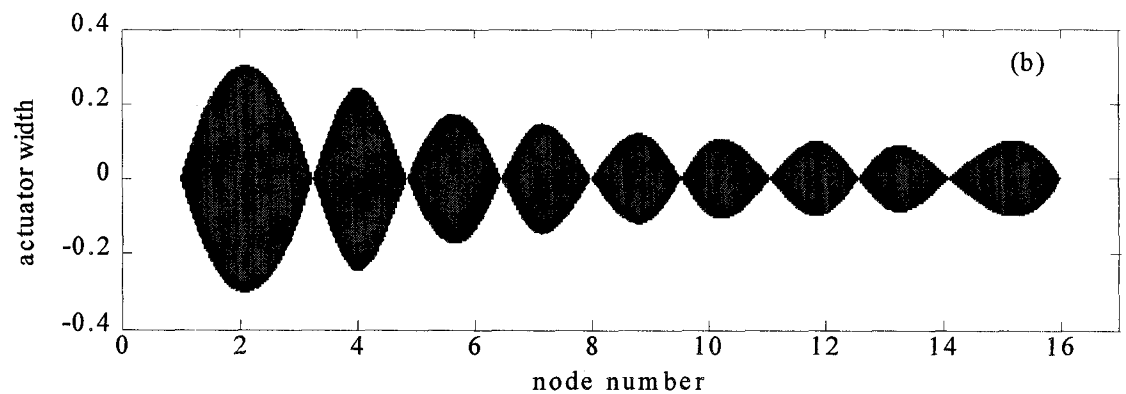
$$B_m = \begin{Bmatrix} 0.01 \\ 0.01 \\ 0.01 \\ 0.01 \\ 0.01 \\ 0.01 \\ 0.01 \\ 0.01 \\ 0.01 \end{Bmatrix}$$

the weighting matrix is obtained from equation as above

Example 2 (cont.)

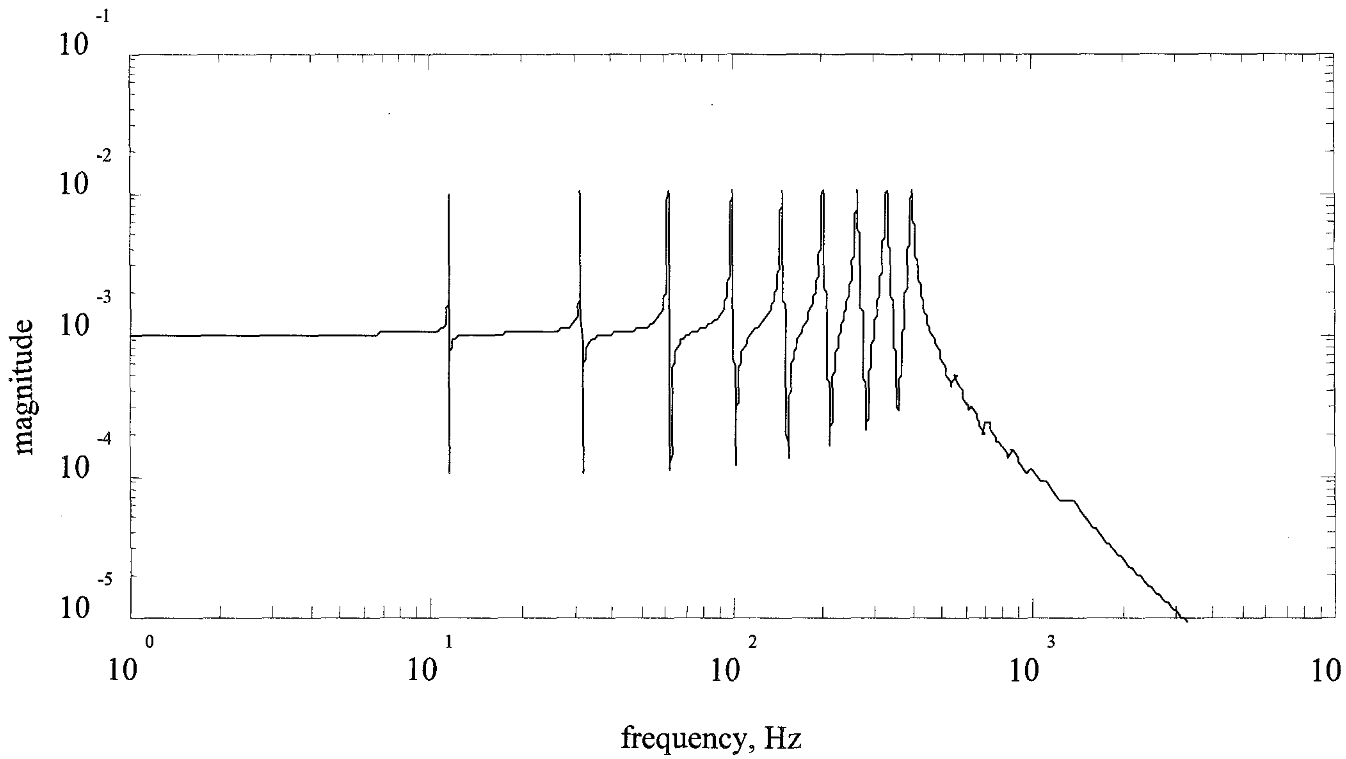
- The resulting gains of the nodal input matrix shown in Fig.a do not follow any particular mode shape. The width of the corresponding piezoelectric actuator that excites all nine modes is shown in b.





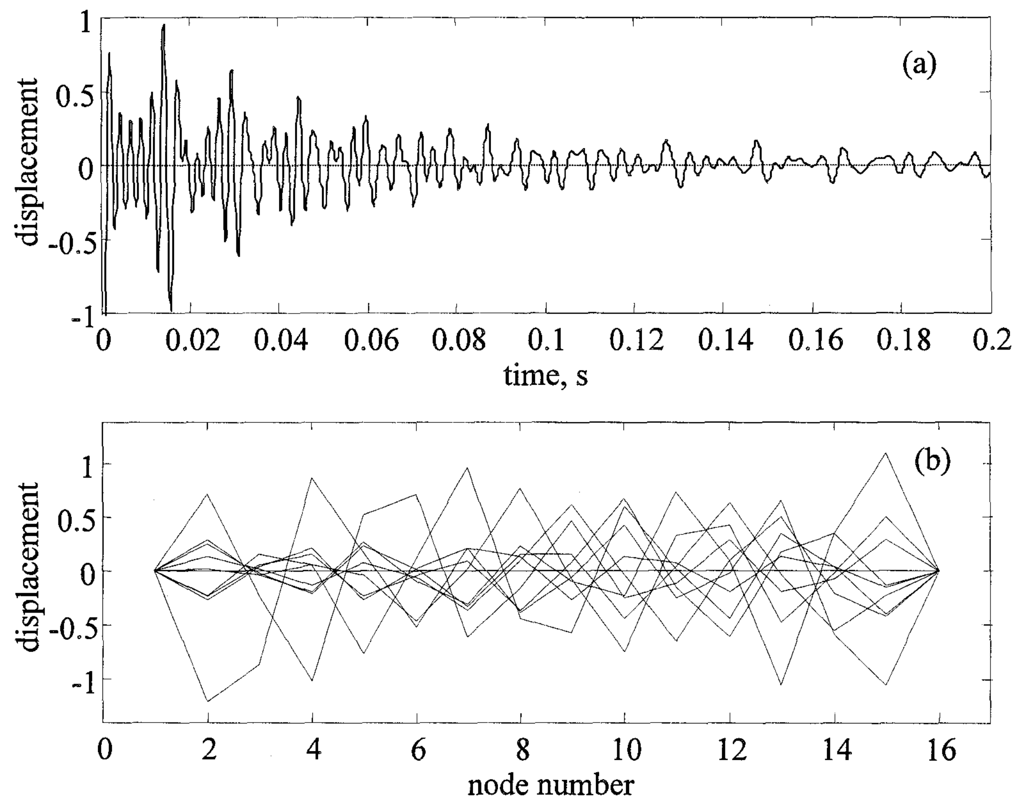
Example 2 (cont.)

The plot of the transfer function shows that all the nine modes are excited, with approximately the same amplitude of 0.01 cm.



Example 2 (cont.)

Fig.a shows the impulse response at node 6, showing nine equally excited modes. Fig.b shows displacements in y direction of all nodes. The rather chaotic pattern of displacement indicates the presence of all 9 modes in the response.



Modal sensors

- The modal sensor determination is similar to the determination of modal actuators. The governing equations

$$C_{mq} = C_{oq}\Phi$$

$$C_{mv} = C_{ov}\Phi$$

If one wants to observe the first mode only one assumes the modal output matrix in the form

$$C_{mq} = \begin{bmatrix} \underset{\uparrow}{1} & 0 & \cdots & 0 \end{bmatrix}$$

first mode

Modal sensors

The corresponding displacement output matrix is

$$C_{oq} = C_{mq} \Phi^{+}$$

For rate sensors:

$$C_{ov} = C_{mv} \Phi^{+}$$

Or alternatively, without pseudoinverse:

$$C_{oq} = C_{mq} M_m^{-1} \Phi^T M$$

$$C_{ov} = C_{mv} M_m^{-1} \Phi^T M$$

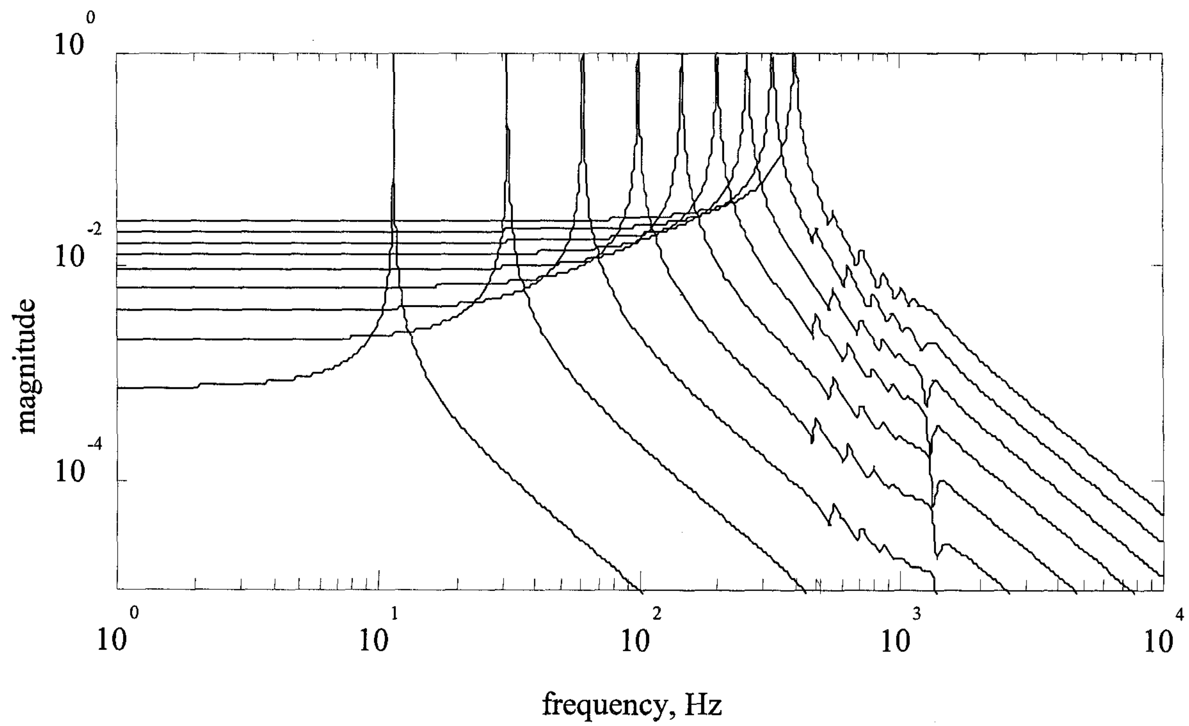
Example 3

- The beam with three vertical force actuators located at nodes 2, 7, and 12 is considered. Find the displacement output matrix such that the first nine modes have equal contribution to the measured output with amplitude 0.01.
- The matrix that excites first nine modes is the unit matrix of dimension 9 of amplitude a_i=0.01, i.e.

$$C_{mq} = \begin{bmatrix} \|c_{m1}\|_2 & 0 & \cdots & 0 \\ 0 & \|c_{m2}\|_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \|c_{mn}\|_2 \end{bmatrix}$$

Example 3 (cont.)

- the magnitudes of the transfer functions for 9 outputs in figure below show that all nine outputs have resonance peak of 0.01.



Example 4

- The beam from with actuators as in Example 3 is considered. Find the nodal rate sensor matrix such that all nine modes but mode 2 contribute equally to the measured output with the amplitude of 0.01.
- The matrix that gives in the equal resonant amplitudes of 0.01 is

$$C_{mv} = 0.01W\begin{bmatrix}1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1\end{bmatrix}$$

second mode

Example 4 (cont.)

- For this matrix the magnitude of the transfer function is shown below, dashed line. It is compared with the magnitude of the transfer function for the output that contains all the 9 modes (solid line). It is easy to notice that the second resonance peak is missing in the plot.